Published by Institute of Physics Publishing for SISSA

Received: *September 23, 2006*
Accepted: *December 4, 2006*
Published: *December 18, 2006*

N = 2 Liouville theory with boundary

Kazuo Hosomichi

Service de Physique Théorique, CEA Saclay
F-91191 Gif sur Yvette, France
E-mail: Kazuo.Hosomichi@cea.fr

Abstract: We study $N = 2$ Liouville theory with arbitrary central charge in the presence of boundaries. After reviewing the theory on the sphere and deriving some important structure constants, we investigate the boundary states of the theory from two approaches, one using the modular transformation property of annulus amplitudes and the other using the bootstrap of disc two-point functions containing degenerate bulk operators. The boundary interactions describing the boundary states are also proposed, based on which the precise correspondence between boundary states and boundary interactions is obtained. The open string spectrum between D-branes is studied from the modular bootstrap approach and also from the reflection relation of boundary operators, providing a consistency check for the proposal.

Keywords: D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction

$N = 2$ Liouville theory has a wide variety of applications in string theory. Although the theory is interacting, the $N = 2$ superconformal symmetry will allow one to compute various structure constants and correlation functions exactly. In the last decade there has been a great progress in the understanding of non-compact, interacting CFTs such as Liouville theories with various supersymmetry[1]–[13]. These recent works have studied the theories by combining the knowledge of the representations of the symmetry algebra together with the Lagrangian description as perturbed free conformal field theories.

A particularly important progress has been made in $N = 0$ and $N = 1$ Liouville theories in the understanding of boundary states or D-branes, where Cardy's construction of boundary states has been successfully applied and we have found various boundary states in consistency with the representation theory of Virasoro or super-Virasoro algebras [8, 9, 12, 13]. For some boundary states the field theory descriptions in terms of boundary interactions have also been provided, whereas the others are interpreted as the theories being realized on the pseudosphere. This big breakthrough was followed by the determination of various exact structure constants on disc [10, 11].

In this paper we try to extend this success to $N = 2$ Liouville theory with boundary. There have been quite a few works [14]–[17] on this theory and also some related works on the dual coset model or the H_3^+ WZW model [20]–[25] along the path explained above. However, there still remain certain confusing issues which we attempt to resolve in the present paper. One source of confusion is the additional periodic direction θ. As we will see, the open and closed string states carry momentum and winding number along θ obeying a certain quantization law, and one has to take a proper account of the quantization law in analyzing the theory. For example, the boundary states in $N = 2$ Liouville theory are classified as A-branes or B-branes according to the choice of boundary conditions on supercurrents, and the momentum/winding number quantization law makes these two families qualitatively very different. In this paper, we are able to take the correct account of the quantization law.

We also study some other difficult problems in $N = 2$ Liouville theory in detail. One of them is related to the property of operators belonging to degenerate representations. The $N = 2$ Liouville theory actually has few properties in common with the less supersymmetric theories. For example, unlike the Liouville theories with less supersymmetry, $N = 2$ Liouville theory does not have a simple strong-weak coupling duality. It instead has as the dual theory the $N = 2$ supersymmetric $SL(2, \mathbb{R})/U(1)$ coset model [26]. One important difference between $N = 2$ theory and $N = 0, 1$ theories is the spectrum of degenerate representations. The degenerate representations of $N = 2$ superconformal algebra are generated by two fundamental degenerate representations with Liouville momentum $j =$

1/2 and $j = k/2$. These two representations are totally different in quality since the former is chiral and the latter is non-chiral, so that they behave very differently under fusion. Another is related to the boundary fermions we are lead to introduce in describing D-branes in terms of boundary interactions. They introduce the Chan-Paton degree of freedom on the boundary and make the properties of boundary operators quite complicated.

This paper is organized as follows. In section 2 we give a rather thorough review of the theory on the sphere, where some OPE coefficients and the reflection coefficients of bulk operators are presented. Section 3 starts the analysis of the theory with boundary, where we find the wave functions for A-branes by analyzing annulus partition functions. We also argue that the similar analysis for B-branes does not work as long as there is a continuous spectrum of closed string states obeying reflection relation. Section 4 gives another derivation of the wave functions which makes use of the Ward identity of disc two-point functions containing degenerate bulk operators. In section 5 we first propose the boundary interactions preserving B-type supersymmetry using the construction well-known in $N = 2$ Landau-Ginzburg models, and then attempt to extend it to A-branes. Using them we calculate some structure constants on the disc and find the relation between boundary couplings and the labels of boundary states. Section 6 analyzes the reflection property of boundary operators, where we find the open string spectrum from the phase of reflection coefficients and check the consistency with the result of modular bootstrap analysis. In section 7 we give some brief concluding remarks. Some useful formulae are recorded in the appendix.

2. N=2 Super-Liouville theory

2.1 Action

The $N = (2, 2)$ superspace has four anti-commuting coordinates $\theta^\pm$ and $\bar\theta^\pm$, and they are related by hermitian conjugation as $(\theta^\pm)^\dagger = \bar\theta^\mp$. The action of $N = 2$ Liouville theory on a flat Euclidean worldsheet is given by

$$I = \frac{1}{8\pi}\int d^2z d\theta^+ d\theta^- d\bar\theta^+ d\bar\theta^- \Phi\bar\Phi + \frac{\mu}{2\pi}\int d^2z d\theta^+ d\bar\theta^+ e^{-\sqrt{\frac{k}{2}}\Phi} + \frac{\bar\mu}{2\pi}\int d^2z d\theta^- d\bar\theta^- e^{-\sqrt{\frac{k}{2}}\bar\Phi}, \tag{2.1}$$

where Φ is a chiral superfield satisfying

$$\begin{aligned}\left(\frac{\partial}{\partial\theta^-} - i\theta^+\partial\right)\Phi &= \left(\frac{\partial}{\partial\bar\theta^-} - i\bar\theta^+\bar\partial\right)\Phi = 0,\\ \left(\frac{\partial}{\partial\theta^+} - i\theta^-\partial\right)\bar\Phi &= \left(\frac{\partial}{\partial\bar\theta^+} - i\bar\theta^-\bar\partial\right)\bar\Phi = 0,\end{aligned} \tag{2.2}$$

and obeying the θ-expansion:

$$\begin{aligned}\Phi &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\bar\theta^+\bar\psi_+ + 2\theta^+\bar\theta^+ F + \cdots\\ \bar\Phi &= \bar\phi + i\sqrt{2}\theta^-\psi_- + i\sqrt{2}\bar\theta^-\bar\psi_- + 2\theta^-\bar\theta^-\bar F + \cdots.\end{aligned} \tag{2.3}$$

JHEP12(2006)061

Writing in components the action becomes, up to total derivatives,

$$
\begin{aligned}
I = \frac{1}{2\pi}\int d^2z \left[\tfrac{1}{2}\partial\phi\bar{\partial}\bar{\phi} + \tfrac{1}{2}\bar{\partial}\phi\partial\bar{\phi} + i\psi_+\bar{\partial}\psi_- + i\bar{\psi}_+\partial\bar{\psi}_- - F\bar{F}\right] \\
-\frac{\mu}{2\pi}\int d^2z(k\psi_+\bar{\psi}_+ - \sqrt{2k}F)e^{-\sqrt{\frac{k}{2}}\phi} - \frac{\bar{\mu}}{2\pi}\int d^2z(k\psi_-\bar{\psi}_- - \sqrt{2k}\bar{F})e^{-\sqrt{\frac{k}{2}}\bar{\phi}}. \quad (2.4)
\end{aligned}
$$

In analyzing supersymmetric field theories, we usually integrate out the auxiliary fields such as F and $\bar{F}$ here to obtain the action written in terms of dynamical fields only. After eliminating the auxiliary fields we obtain the exponential potential for the real part of ϕ, so the theory describes the dynamics of strings in the presence of Liouville-like potential wall. However, to make use of the calculational techniques in CFT we would rather not integrate over $F, \bar{F}$ first. Generic vertex operators are therefore local functionals of dynamical fields as well as $F, \bar{F}$. Since the auxiliary fields give contact interaction, under some restriction on the momenta of vertex operators we may simply neglect their contributions to the correlation functions. We can also see that by simply putting $F = \bar{F} = 0$ the action reduces to the system of free fields with $N = 2$ superconformal symmetry perturbed by exponential operators

$$
\begin{aligned}
I = \frac{1}{2\pi}\int d^2z \left[\partial\rho\bar{\partial}\rho - \sqrt{\tfrac{2}{k}}\tfrac{R\rho}{4} + \partial\theta\bar{\partial}\theta + \psi_+\bar{\partial}\psi_- + \bar{\psi}_+\partial\bar{\psi}_-\right] \\
+\frac{ik\mu}{2\pi}\int d^2z\psi_+\bar{\psi}_+e^{-\sqrt{\frac{k}{2}}\phi} + \frac{ik\bar{\mu}}{2\pi}\int d^2z\psi_-\bar{\psi}_-e^{-\sqrt{\frac{k}{2}}\bar{\phi}}. \quad (2.5)
\end{aligned}
$$

Here we introduced the real bosons ρ, θ by $\phi = \rho + i\theta$, and redefined the fermions so as to satisfy the canonical OPEs

$$
\rho(z)\rho(w) \sim \theta(z)\theta(w) \sim -\ln|z-w|^2, \quad \psi_+(z)\psi_-(w) \sim \frac{2}{z-w}, \quad (2.6)
$$

The system of free fields defined by the first line of the action (2.5) represents a $N = 2$ superconformal algebra with the central charge $\hat{c} = \frac{c}{3} = 1 + \frac{2}{k}$:

$$
\begin{aligned}
T &= -\tfrac{1}{2}(\partial\rho\partial\rho + \sqrt{\tfrac{2}{k}}\partial^2\rho + \partial\theta\partial\theta) - \tfrac{1}{4}(\psi_+\partial\psi_- + \psi_-\partial\psi_+), \\
\sqrt{2}T_F^+ &= i\psi_+\partial\bar{\phi} + i\sqrt{\tfrac{2}{k}}\partial\psi_+, \\
\sqrt{2}T_F^- &= i\psi_-\partial\phi + i\sqrt{\tfrac{2}{k}}\partial\psi_-, \\
J &= \tfrac{1}{2}\psi_+\psi_- + i\sqrt{\tfrac{2}{k}}\partial\theta, \quad (2.7)
\end{aligned}
$$

and the interaction terms in the second line commute with these currents. From this we see that by simply dropping the auxiliary fields we still have a superconformal symmetry. As in the $N = 0$ and $N = 1$ Liouville theories, the interaction terms screen the momentum along ρ or θ directions and therefore the momentum along these directions does not conserve. For this reason we sometimes refer to these interaction terms as *screening operators*. The easiest way to calculate various quantities is therefore to restrict first the momenta of vertex operators so that the contact terms do not contribute, and then make the analytic continuation in the momenta. Some quantities are expressed as correlators of free fields with some screening operators inserted. As we will illustrate later in a few examples, the role of auxiliary fields is to cancel some of the divergences that arise in naive screening integral expressions. For more detailed discussions on these matters, see [27−29].

From the viewpoint of free CFT perturbed by screening operators, there is another screening operator which can be written as a D-term invariant

$$\begin{aligned}&\int d^2z d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \exp\left(-\tfrac{1}{\sqrt{2k}}(\Phi+\bar{\Phi})\right)\\ &= \sqrt{\tfrac{2}{k}}\partial\bar{\partial}\rho e^{-\sqrt{\frac{2}{k}}\rho} + \tfrac{i}{k}\left\{\psi_+\bar{\partial}\psi_- - \bar{\partial}\psi_+\psi_- + \bar{\psi}_+\partial\bar{\psi}_- - \partial\bar{\psi}_+\bar{\psi}_-\right\} e^{-\sqrt{\frac{2}{k}}\rho}\\ &\quad+\tfrac{1}{k^2}\left\{-2kF\bar{F} + \sqrt{2k}\psi_+\bar{\psi}_+\bar{F} + \sqrt{2k}\psi_-\bar{\psi}_-F\right\} e^{-\sqrt{\frac{2}{k}}\rho}\\ &\quad+\tfrac{1}{k^2}(\psi_+\psi_- - \sqrt{2k}\partial\theta)(\bar{\psi}_+\bar{\psi}_- - \sqrt{2k}\bar{\partial}\theta)e^{-\sqrt{\frac{2}{k}}\rho}.\end{aligned} \tag{2.8}$$

The first two lines are neglected since they only give contact interactions, and the last line gives, after canonical normalization of fermions, the following screening operator:

$$(\psi_+\psi_- - i\sqrt{2k}\partial\theta)(\bar{\psi}_+\bar{\psi}_- - i\sqrt{2k}\bar{\partial}\theta)e^{-\sqrt{\frac{2}{k}}\rho}. \tag{2.9}$$

It is useful to bosonize the fermions in terms of a compact boson H:

$$\psi_\pm = \sqrt{2}e^{\pm iH_L}, \quad \bar{\psi}_\pm = \sqrt{2}e^{\pm iH_R}. \tag{2.10}$$

where the suffices L, R indicate the holomorphic and anti-holomorphic parts. The two screening operators that were in the original action are rewritten as follows:

$$\mu S + \bar{\mu}\bar{S} = -\frac{k\mu}{\pi}\int d^2z e^{-\sqrt{\frac{k}{2}}\phi + iH} - \frac{k\bar{\mu}}{\pi}\int d^2z e^{-\sqrt{\frac{k}{2}}\bar{\phi} - iH}. \tag{2.11}$$

2.2 Vertex operators

As bulk operators which are inserted in the interior of the worldsheet, we mainly consider those of the form

$$V^{j(s,\bar{s})}_{m,\bar{m}} \equiv \exp\left[\sqrt{\tfrac{2}{k}}\left\{j\rho + i(m+s)\theta_L + i(\bar{m}+\bar{s})\theta_R\right\} + isH_L + i\bar{s}H_R\right]. \tag{2.12}$$

The labels $(s, \bar{s})$ determines the monodromy of fermions $(\psi_\pm, \bar{\psi}_\pm)$ around this operator. In particular, NS sector corresponds to $s, \bar{s} \in \mathbb{Z}$ and the R sector to $s, \bar{s} \in \mathbb{Z} + \frac{1}{2}$. These labels are also regarded as the amounts of *spectral flow* explained later.

For the correlators of these vertex operators to be calculable perturbatively, the interaction terms must be single-valued around them. This gives the constraint $m - \bar{m} \in \mathbb{Z}$. Hereafter we shall restrict our attention to such operators and call them *perturbatively well-defined*. Note that this is not enough to ensure the mutual locality of these vertex operators.

The θ corresponds to the phase of the chiral field $\exp(-\sqrt{k/2}\Phi)$, and it has the periodicity $2\pi\sqrt{2/k}$. The periodicity can also be read off from the behavior of θ around a perturbatively well-defined operator,

$$\theta(z)V^j_{m\bar{m}}(0) \sim -i\sqrt{\tfrac{2}{k}}(m\ln z + \bar{m}\ln\bar{z})V^j_{m\bar{m}}(0), \tag{2.13}$$

and $m-\bar{m}$ therefore corresponds to the winding number along θ-direction. The θ-momentum $\frac{1}{\sqrt{2k}}(m+\bar{m})$ should also be quantized in unit of $\sqrt{\frac{k}{2}}$. We thus have the quantization law

$$m, \bar{m} = \frac{kn \pm w}{2}. \quad (n, w \in \mathbb{Z}) \tag{2.14}$$

Physical spectrum of *closed string states* i.e. the states on a circle, should obey this condition. But we sometimes consider bulk operators not satisfying this in the calculation of correlators on the sphere or the disc. The same argument for operators with nonzero $s, \bar{s}$ leads to the quantization law

$$m - \bar{m} \in \mathbb{Z}, \quad m + s + \bar{m} + \bar{s} \in k\mathbb{Z}. \tag{2.15}$$

2.3 Representations of N = 2 superconformal algebra

The $N = 2$ superconformal algebra is generated by the currents $T, T_F^{\pm}$ and J satisfying the OPEs

$$\begin{array}{llll} T(z)T(0) \sim \frac{3\hat{c}}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, & T(z)J(0) \sim \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}, \\ T(z)T_F^{\pm}(0) \sim \frac{3T_F(0)}{2z^2} + \frac{\partial T_F(0)}{z}, & J(0)T_F^{\pm}(0) \sim \pm\frac{T_F^{\pm}(0)}{z}, \\ T_F^{+}(z)T_F^{-}(0) \sim \frac{2\hat{c}}{z^3} + \frac{2J(0)}{z^2} + \frac{2T(0)}{z} + \frac{\partial J(0)}{z}, & J(z)J(0) \sim \frac{\hat{c}}{z^2}. \end{array} \tag{2.16}$$

If we define their modes as follows

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad T_F^{\pm}(z) = \sum_{r\in\mathbb{Z}\pm\alpha+1/2} G_r^{\pm} z^{-n-3/2}, \quad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \tag{2.17}$$

they obey the (anti-)commutation relations

$$\begin{array}{ll} [L_m, L_n] = (m-n)L_{m+n} + \frac{\hat{c}}{4}(m^3 - m)\delta_{m+n,0}, & [L_m, J_n] = -nJ_{m+n}, \\ [L_m, G_n^{\pm}] = \left(\frac{m}{2} - n\right) G_{m+n}^{\pm}, & [J_m, G_n^{\pm}] = \pm G_{m+n}^{\pm}, \\ \{G_m^{+}, G_n^{-}\} = 2L_{m+n} + (m-n)J_{m+n} + \hat{c}(m^2 - \frac{1}{4})\delta_{m+n,0}, & [J_m, J_n] = \hat{c}m\delta_{m+n,0}. \end{array} \tag{2.18}$$

NS and R algebras are labelled by $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively. For *open string states*, i.e. states on a strip, we will have to consider other algebras labelled by arbitrary real α. Such algebras are related to one another by spectral flow:

$$\begin{aligned} U^{-\alpha}L_nU^{\alpha} &= L_n + \alpha J_n + \frac{\alpha^2\hat{c}}{2}\delta_{m+n,0}, \\ U^{-\alpha}G_n^{\pm}U^{\alpha} &= G_{n\pm\alpha}^{\pm}, \\ U^{-\alpha}J_nU^{\alpha} &= J_n + \alpha\hat{c}\delta_{n,0}. \end{aligned} \tag{2.19}$$

The spectral flows labelled by $\alpha \in \mathbb{Z}$ are automorphisms of the $N = 2$ superconformal algebra.

JHEP12(2006)061

To any vertex operator there corresponds a representation of superconformal algebra. Let us take as an example the bulk operator introduced in the previous subsection and focus on its left-moving part:

$$V_m^{j(s)}(z) = \exp\left[\sqrt{\tfrac{2}{k}}\,\{j\rho_L + i(m+s)\theta_L\} + isH_L\right]. \tag{2.20}$$

It corresponds to the state $|j,m,s\rangle$ which has the L_0 and J_0 eigenvalues (h,Q):

$$h = \frac{(m+s)^2 - j(j+1)}{k} + \frac{s^2}{2}, \quad Q = \frac{2(m+s)}{k} + s, \tag{2.21}$$

and is annihilated by $G^+_{r\geq\frac{1}{2}-s}$ and $G^-_{r\geq\frac{1}{2}+s}$. The operators with $s=0$ correspond to NS sector primary states, and s represents the amount of spectral flow:

$$|j,m,s\rangle = U^s|j,m,0\rangle. \tag{2.22}$$

The action of supercurrents on them reads

$$T_F^{\pm}(z)V_m^{j(s)}(0) \sim -i\sqrt{\tfrac{2}{k}}(j\pm m)z^{\pm s-1}V_{m\mp 1}^{j(s\pm 1)}(0). \tag{2.23}$$

The states with $j=\mp m$ are annihilated by $G^{\pm}_{\pm s-\frac{1}{2}}$. They are (anti-)chiral primary states spectral flowed by s units. The above formula also shows that the two highest weight representations are related by an integer spectral flow when their m labels differ by an integer.

2.3.1 Degenerate representations

As can be read off from the determinant formula of [30, 31], the Verma module of NS algebra labelled by conformal weight h and R-charge Q contains a null vector when

$$f_{r,s}(h,Q) \equiv 2(\hat{c}-1)h - Q^2 - \tfrac{1}{4}(\hat{c}-1)^2 + \tfrac{1}{4}\{(\hat{c}-1)r + 2s\}^2 = 0 \quad (r,s\in\mathbb{Z}_{>0}), \tag{2.24}$$

and the null vector appears at level rs. It follows from this that NS vertex operator V_m^j corresponds to a degenerate representation when

$$2j+1 = \pm(r+ks)\,. \quad (r,s\in\mathbb{Z}_{>0}) \tag{2.25}$$

The most fundamental degenerate operators within this category are those with $(r,s) = (1,1)$ or $j=\frac{k}{2}$. According to [30] the determinant also vanishes when

$$g_p(h,Q) \equiv 2h - 2pQ + (\hat{c}-1)(p^2 - \tfrac{1}{4}) = 0 \quad (p\in\mathbb{Z}+\tfrac{1}{2}). \tag{2.26}$$

In the simplest case $p=\pm\frac{1}{2}$ we have $2h=\pm Q$. For generic p we can easily find a null vector χ of the form

$$\begin{aligned}(p>0)\quad |\chi\rangle &= G^+_{-p}\cdots G^+_{-3/2}G^+_{-1/2}|h,Q\rangle,\\ (p<0)\quad |\chi\rangle &= G^-_{p}\cdots G^-_{-3/2}G^-_{-1/2}|h,Q\rangle,\end{aligned} \tag{2.27}$$

so they are chiral representations spectral flowed by $p - \frac{1}{2}$ units, or anti-chiral representations spectral flowed by $p + \frac{1}{2}$ units. In particular, the representation has two null vectors[1] if (h, Q) are such that there are two different values of p satisfying (2.26). In terms of the labels (j, m) the condition is simply

$$(j \pm m \in \mathbb{Z}_{\geq 0}) \quad \text{or} \quad (-j - 1 \pm m \in \mathbb{Z}_{\geq 0}). \tag{2.28}$$

Though we will not explain in detail, classification of representations of $SL(2,\mathbb{R})$ current algebra at level $k+2$ is also known, and degenerate representations appear precisely in the same manner as those of $N = 2$ superconformal algebra. For example, The representations with two null vectors correspond to finite dimensional representations of $SL(2,\mathbb{R})$.

2.3.2 Unitary representations

In [30] the conditions for unitary representations were also given. For $k > 0$ there are two classes of unitary representations of NS algebra. The first ones satisfy

$$g_p(h, Q) \geq 0 \quad \text{or} \quad (j + \tfrac{1}{2})^2 \leq (m - p)^2 \quad \text{for all} \;\; p \in \mathbb{Z} + \tfrac{1}{2}. \tag{2.29}$$

The representations with $j \in -\frac{1}{2} + i\mathbb{R}$ are therefore all unitary irrespective of the value of m. There are also some unitary representations with $-1 < j < 0$ in this class, depending on the value of m. The second ones satisfy

$$g_p(h, Q) = 0, \quad g_{p+\mathrm{sgn}(p)}(h, Q) < 0 \;\; \text{and} \;\; f_{1,1}(h, Q) \geq 0. \tag{2.30}$$

In terms of j, m this condition becomes, up to $j \leftrightarrow -j - 1$ equivalence,

$$-\tfrac{k}{2} - 1 < j < -\tfrac{1}{2} \;\; \text{and} \;\; \pm m \in j - \mathbb{Z}_{\geq 0} \tag{2.31}$$

Hereafter we will use the term discrete/continuous representations for these two classes of representations. The bound for j is called the *unitarity bound*, and we actually expect a little more stringent bound for discrete series from recent works. This can also be understood from the reflection relation for chiral operators sending j to $-j - 1 - \frac{k}{2}$, which we will explain later.

2.4 Perturbed linear dilaton CFT

The $N = 2$ Liouville theory can be analyzed as a linear dilaton theory (free CFT) with exponential type perturbations. As was found in [1], the correlators of such theories are calculable simply as Wick contractions of free CFT with a certain number of screening operators inserted.

The simplest example of such theories is the bosonic Liouville theory, defined by the action

$$I = \frac{1}{8\pi}\int d^2x\sqrt{g}\left[g^{mn}\partial_m\phi\partial_n\phi + \sqrt{2}QR\phi + 8\pi\mu e^{\sqrt{2}b\phi}\right], \quad Q = b + b^{-1}. \tag{2.32}$$

[1]Note that the relevant Verma module contains more than two null vectors. For example, the Verma module over the NS ground state has three null vectors, $G^{\pm}_{-1/2}|0\rangle$ and $L_{-1}|0\rangle$.

JHEP12(2006)061

We are interested in the correlators typically of the form

$$\langle\prod_i e^{\sqrt{2}\alpha_i\phi(z_i)}\rangle = \int \mathcal{D}\phi e^{-I}\prod_i e^{\sqrt{2}\alpha_a\phi(z_i)}. \tag{2.33}$$

Extracting the dependence on the zero-mode of ϕ we find, on a worldsheet with genus g, the following integral:

$$\begin{aligned}\int d\phi_0 \exp\left[\sqrt{2}\{\sum_i \alpha - Q(1-g)\}\phi_0 - e^{\sqrt{2}b\phi_0}\mu\int d^2x e^{\sqrt{2}b(\phi-\phi_0)}\right]\\ = \frac{\Gamma(-N)}{\sqrt{2}b}\left\{\mu\int d^2x e^{\sqrt{2}b(\phi-\phi_0)}\right\}^N. \quad (bN = Q(1-g) - \sum\alpha_i)\end{aligned} \tag{2.34}$$

N is the number of screening operators necessary to cancel the momentum carried by vertices and also by the background with genus g. Since the integration over non-zero mode of ϕ is equivalent to taking the Wick contraction using the free correlator, we obtain the following formal expression

$$\langle\prod_i e^{\sqrt{2}\alpha_i\phi(z_i)}\rangle = \frac{\Gamma(-N)}{\sqrt{2}b}\langle\prod_i e^{\sqrt{2}\alpha_i\phi(z_i)}(\mu S)^N\rangle_{\text{free}}, \quad \mu S = \mu\int d^2x e^{\sqrt{2}b\phi}. \tag{2.35}$$

This shows that the correlator diverges when the total momentum of vertices and the background can be cancelled by a non-negative integer insertion of screening operators,

$$Q(1-g) - \sum\alpha_i \in b\mathbb{Z}_{\geq 0}, \tag{2.36}$$

and the residue of such divergences is given by the Wick contraction of free fields. More explicitly, by rewriting the Gamma function as a sum of simple poles we obtain an expression

$$\langle\prod_i e^{\sqrt{2}\alpha_i\phi(z_i)}\rangle \simeq \sum_{n\geq 0}\frac{1}{\sqrt{2}}\frac{1}{nb + \Sigma\alpha_i - Q(1-g)}\langle\prod_i e^{\sqrt{2}\alpha_i\phi(z_i)}\frac{(-\mu S)^n}{n!}\rangle_{\text{free}}, \tag{2.37}$$

which well approximates the behavior of correlators near the poles $Q(1-g) - \sum\alpha_i \in b\mathbb{Z}_{\geq 0}$.

The above argument applies also to the $N = 2$ Liouville theory. Let us consider the correlator

$$\langle\prod_i V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i)\rangle = \int \mathcal{D}\rho\mathcal{D}\theta\mathcal{D}\psi e^{-I}\prod_i V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i). \tag{2.38}$$

In this theory we have two screening operators (with couplings μ and $\bar{\mu}$) in the defining action, so let us expand into power series in μ and then integrate over the zero mode of ρ. Then we finally obtain

$$\langle\prod_i V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i)\rangle \simeq \sqrt{\frac{k}{2}}\sum_{n,\bar{n}\geq 0}\frac{1}{n!\bar{n}!}\frac{\langle\prod_i V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i)(-\mu S)^n(-\bar{\mu}\bar{S})^{\bar{n}}\rangle_{\text{free}}}{-\sum_i j_i - 1 + g + \frac{k}{2}(n+\bar{n})}, \tag{2.39}$$

which well approximates the behavior of correlators near the poles $\sum j_i + 1 - g \in \frac{k}{2}\mathbb{Z}_{\geq 0}$.

2.4.1 Three-point function

Using this formula we calculate the three-point function of operators $V^{j(s,\bar{s})}_{m\bar{m}}$. We first evaluate the residues of the poles corresponding to integer insertions of screening operators, and then obtain the correlator by some kind of *extrapolation*. Similar calculations were performed for bosonic Liouville theory in [2, 3] and $N = 1$ Liouville theory in [5, 6].

The residues of the poles in (2.39) are given by the Wick contraction:

$$\frac{1}{n!\bar{n}!}\langle\prod_{i=1}^{3} V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i)(-\mu S)^n(-\bar{\mu}\bar{S})^{\bar{n}}\rangle_{\text{free}}. \tag{2.40}$$

In order to account for the anti-commutativity of Grassmann odd quantities, we need to include cocycle factors in doing the contraction. We therefore assign the factor

$$\exp\left(\frac{i\pi}{k}((m_1+s_1)(\bar{m}_2+\bar{s}_2)-(m_2+s_2)(\bar{m}_1+\bar{s}_1))+\frac{i\pi}{2}(s_1\bar{s}_2-s_2\bar{s}_1)\right) \tag{2.41}$$

in contracting the product $V^{j_1(s_1\bar{s}_1)}_{m_1\bar{m}_1}V^{j_2(s_2\bar{s}_2)}_{m_2\bar{m}_2}$. This ensures that the operators are simply commuting or anti-commuting according to their Grassmann parity if they satisfy (2.15) as well as $s,\bar{s}\in\mathbb{Z}$. In particular, the positions of screening operators do not matter in calculating correlation functions. Note also that the momentum conservation of linear dilaton theory requires

$$\Sigma j_a+1=\tfrac{k}{2}(n+\bar{n}),\quad \Sigma m_a=\Sigma\bar{m}_a=(1+\tfrac{k}{2})(n-\bar{n}),\quad \Sigma s_a=\Sigma\bar{s}_a=-n+\bar{n}. \tag{2.42}$$

After the Wick contraction we encounter the integral

$$\begin{aligned}I\ =\ &\frac{(-)^{(n-\bar{n})(m_1-\bar{m}_1)}}{n!\bar{n}!}\int\prod_{i=1}^{n}d^2z_iz_i^{j_1-m_1}\bar{z}_i^{j_1-\bar{m}_1}(1-z_i)^{j_2-m_2}(1-\bar{z}_i)^{j_2-\bar{m}_2} \qquad (2.43)\\ &\times\prod_{\hat{i}=1}^{\bar{n}}d^2z_{\hat{i}}z_{\hat{i}}^{j_1+m_1}\bar{z}_{\hat{i}}^{j_1+\bar{m}_1}(1-z_{\hat{i}})^{j_2+m_2}(1-\bar{z}_{\hat{i}})^{j_2+\bar{m}_2}\prod_{i<j}|z_{ij}|^2\prod_{\hat{i}<\hat{j}}|z_{\hat{i}\hat{j}}|^2\prod_{i,\hat{j}}|z_{i\hat{j}}|^{-2k-2},\end{aligned}$$

which is calculated using the formula[2] in [32]. It is non-vanishing only when $n-\bar{n}=\pm1$ or 0.

$$\begin{aligned}&(n=\bar{n}+1)\\ I\ =\ &\pi^{2n-1}\left[\prod_{r=1}^{n-1}\gamma(-rk)\gamma(1+2j_1-rk)\gamma(1+2j_2-rk)\gamma(1+2j_3-rk)\right]F_-(j_a,m_a,\bar{m}_a),\\ F_\pm(j_a,m_a,\bar{m}_a)\ =\ &(-)^{m_2-\bar{m}_2}\frac{\Gamma(1+j_1\pm m_1)\Gamma(1+j_2\pm m_2)\Gamma(1+j_3\pm m_3)}{\Gamma(-j_1\mp\bar{m}_1)\Gamma(-j_2\mp\bar{m}_2)\Gamma(-j_3\mp\bar{m}_3)},\\ &(n=\bar{n})\\ I\ =\ &\pi^{2n}\left[\prod_{r=1}^{n-1}\gamma(-rk)\gamma(1+2j_1-rk)\gamma(1+2j_2-rk)\gamma(1+2j_3-rk)\right]F(j_a,m_a,\bar{m}_a),\\ F(j_a,m_a,\bar{m}_a)=\ &\pi^{-2}\int d^2zd^2wz^{j_1-m_1}\bar{z}^{j_1-\bar{m}_1}(1-z)^{j_2-m_2}(1-\bar{z})^{j_2-\bar{m}_2}\\ &\times w^{j_1+m_1}\bar{w}^{j_1+\bar{m}_1}(1-w)^{j_2+m_2}(1-\bar{w})^{j_2+\bar{m}_2}|z-w|^{-4-2(j_1+j_2+j_3)}.\end{aligned} \tag{2.44}$$

[2]Some typos by unnecessary sign factors there are corrected here.

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. Using $k = b^{-2}$ and the special function Υ introduced in [3] (see the appendix for the definition) we can rewrite the products of γ functions as follows:

$$
\begin{aligned}
&n = \bar{n} + 1 = b^2(j_{1+2+3}+1) + \tfrac{1}{2} : \\
&I = F_-(j_a, m_a, \bar{m}_a) \frac{\pi^{2n-1} b^{2(1+k)(n-1)} \Upsilon'(0)\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\Upsilon'(\frac{1}{b}(1-n))\Upsilon(\frac{1}{2b}+bj_{1-2-3})\Upsilon(\frac{1}{2b}+bj_{2-3-1})\Upsilon(\frac{1}{2b}+bj_{3-1-2})}, \\
&n = \bar{n} = b^2(\Sigma j_a + 1) : \\
&I = F(j_a, m_a, \bar{m}_a) \frac{\pi^{2n} b^{2(n-1)} \Upsilon'(0)\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\Upsilon'(\frac{1}{b}(1-n))\Upsilon(\frac{1}{b}+bj_{1-2-3})\Upsilon(\frac{1}{b}+bj_{2-3-1})\Upsilon(\frac{1}{b}+bj_{3-1-2})}.
\end{aligned}
\tag{2.45}
$$

The derivative of Υ function appears because we re-wrote a product of γ functions by the ratio of Υ functions both are vanishing. The function $\Upsilon(\frac{1}{b}(1-n))$ vanishes precisely at the poles of the three-point function appearing in (2.39). So we combine the sum over poles into a single function using the *extrapolation*

$$
\sum_{a\in\{\text{simple zeroes of } F(x)\}} \frac{1}{(x-a)F'(a)} \simeq \frac{1}{F(x)}. \tag{2.46}
$$

Combining with other factors we obtain

$$
\begin{aligned}
&\langle \textstyle\prod_{a=1}^3 V^{j_a(s_a\bar{s}_a)}_{m_a\bar{m}_a}(z_a)\rangle \\
&= \prod_{\{abc\}=\{321\},\{312\},\{213\}} z_{ab}^{h_c-h_a-h_b} \bar{z}_{ab}^{\bar{h}_c-\bar{h}_a-\bar{h}_b} e^{\frac{i\pi}{k}\{(m_a+s_a)(\bar{m}_b+\bar{s}_b)-(m_b+s_b)(\bar{m}_a+\bar{s}_a)\}+\frac{i\pi}{2}(s_a\bar{s}_b-s_b\bar{s}_a)} \\
&\quad\times \left\{\delta^2(\Sigma_i m_i - \tfrac{k\hat{c}}{2})\delta^2(\Sigma_i s_i+1)F_-(j_a,m_a,\bar{m}_a)D_- \right. \\
&\quad\left. + \delta^2(\Sigma_i m_i + \tfrac{k\hat{c}}{2})\delta^2(\Sigma_i s_i-1)F_+(j_a,m_a,\bar{m}_a)D_+ + \delta^2(\Sigma_i m_i)\delta^2(\Sigma_i s_i)F(j_a,m_a,\bar{m}_a)D_0 \right\},
\end{aligned}
$$

$$
\begin{aligned}
D_\pm &= \frac{(\nu b^{2-2b^2})^{j_{1+2+3}+1}\Upsilon'(0)\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\sqrt{2}b^{1+k}\Upsilon(\frac{1}{2b}-b(j_{1+2+3}+1))\Upsilon(\frac{1}{2b}+bj_{1-2-3})\Upsilon(\frac{1}{2b}+bj_{2-3-1})\Upsilon(\frac{1}{2b}+bj_{3-1-2})}, \\
D_0 &= \frac{(\nu b^{-2b^2})^{j_{1+2+3}+1}\Upsilon'(0)\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\sqrt{2}b^2\Upsilon(\frac{1}{b}-b(j_{1+2+3}+1))\Upsilon(\frac{1}{b}+bj_{1-2-3})\Upsilon(\frac{1}{b}+bj_{2-3-1})\Upsilon(\frac{1}{b}+bj_{3-1-2})},
\end{aligned}
\tag{2.47}
$$

where we set $\mu = \bar{\mu} = \nu^{\frac{k}{2}}$ using the translation along θ-direction.

The three-point structure constants have much more poles than the path-integral formula predicts. Some of the poles can be accounted for by incorporating another screening operator of D-term type given in (2.8). The poles in the ρ-momentum space at

$$
j_1 + j_2 + j_3 + 1 = \tfrac{k}{2}(n+\bar{n}) + m \qquad (n, \bar{n}, m \in \mathbb{Z}_{\geq 0}) \tag{2.48}
$$

can be explained in this way. Note that in bosonic Liouville theory we meet a similar situation, where we have two screening operators although only one of them is present in the defining action. The two screening operators there are transformed to each other by $b \leftrightarrow \frac{1}{b}$ flip, so it follows that the theory has the strong/weak coupling ($b \leftrightarrow \frac{1}{b}$) duality.

All the other poles are related to the ones explained above by reflection relations which are explained later.

2.4.2 Contact interactions

Let us make some comments here on the role of the auxiliary fields which we have neglected in the calculations above. Before doing this, we would like to note that at the stage of evaluating the screening integral using the formula (2.44) we have already assumed the analyticity. Otherwise the validity of the formula for correlators would be very restricted in the momentum space, because any integrals which look like

$$\int d^2x x^{\alpha}\bar{x}^{\bar{\alpha}}\cdots \quad (\alpha-\bar{\alpha}\in\mathbb{Z}) \tag{2.49}$$

diverge around $x\sim 0$ when $\max(\alpha,\bar{\alpha})\le -1$. After the analytic continuation, one can see the divergences as poles of the functions like $\Gamma(1+\alpha)$ or $\Gamma(1+\bar{\alpha})$. Assuming analyticity, away from such poles in the momentum space we are allowed to do all the naive operation such as partial integration. From this viewpoint, it would not be so bad to simply discard the contact interactions, calculate correlators in a region of momentum space where contact terms are negligible and then analytically extend. But of course one can treat the contact terms honestly and find that they play the role of *ensuring the analyticity.*

As will be illustrated below, the auxiliary fields *cancel* some of the divergences in correlators. We begin by recalling that, when a correlator contains a primary operator $e^{-(p\phi+\bar{p}\bar{\phi})}$, the screening integral behaves around it as (neglecting coefficients)

$$\mu S\cdot e^{-(p\phi+\bar{p}\bar{\phi})}(0)\sim\int d^2z|z|^{-2\sqrt{2k}\bar{p}},\qquad \bar{\mu}\bar{S}\cdot e^{-(p\phi+\bar{p}\bar{\phi})}(0)\sim\int d^2z|z|^{-2\sqrt{2k}p}. \tag{2.50}$$

So there are no divergences from integrals over origin as long as $p,\bar{p}<\sqrt{1/2k}$. From the superconformal symmetry, it is natural to expect that the screening integrals containing $e^{-(p\phi+\bar{p}\bar{\phi})}$ or *any of its descendants* are finite for those values of $p,\bar{p}$. Now, let us take a descendant:

$$\int d\theta^+ d\bar{\theta}^+ e^{-(p\Phi+\bar{p}\bar{\Phi})}=2p(F+ip\psi_+\bar{\psi}_+)e^{-(p\phi+\bar{p}\bar{\phi})}, \tag{2.51}$$

put at $z=0$ in a certain correlator. We should then consider the divergences of screening integrals around this operator. One finds the singular behavior of μS becomes milder, whereas that of $\bar{\mu}\bar{S}$ gets stronger, since the latter contains the fermions with opposite charge:

$$\bar{\mu}\bar{S}=\frac{\bar{\mu}}{2\pi}\int d^2z\left(ik\psi_-\bar{\psi}_-+\sqrt{2k}\bar{F}\right)e^{-\sqrt{\frac{k}{2}}\bar{\phi}}. \tag{2.52}$$

So the $\bar{\mu}\bar{S}$-integral is apparently finite only for negative p.

Let us show that, through suitable regularization, the $\bar{\mu}\bar{S}$-integral is actually finite for p slightly above zero due to the cancellation between contact and non-contact interactions. The screening integral is the sum of a non-contact and a contact terms, both of which are divergent for positive $\bar{p}$. In order to compare the two divergences, let us introduce the following regularization. First, reguralize the non-contact term by cutting off the integration domain by a hole of radius ϵ. The non-contact interaction is then evaluated as follows:

$$\frac{ik\bar{\mu}}{2\pi}\int d^2z\psi_-\bar{\psi}_-e^{-\sqrt{\frac{k}{2}}\bar{\phi}}(z)\times(2ip^2)\psi_+\bar{\psi}_+e^{-(p\phi+\bar{p}\bar{\phi})}(0)$$

$$\sim \frac{4kp^2\bar{\mu}}{\pi}\int_{|z|\geq\epsilon} d^2z|z|^{-2-2\sqrt{2k}p} = 2\sqrt{2k}p\bar{\mu}\epsilon^{-2\sqrt{2k}p}. \tag{2.53}$$

Second, separate F and $e^{-(p\phi+\bar{p}\bar{\phi})}$ by spreading F along the boundary of the same hole. The contact interaction is also regularized, and it precisely cancels with the non-contact interaction

$$\frac{\bar{\mu}}{2\pi}\int d^2z\sqrt{2k}\bar{F}e^{-\sqrt{\frac{k}{2}}\bar{\phi}}(z)\times 2p\oint_\epsilon \frac{dx}{2\pi ix}F(x)e^{-(p\phi+\bar{p}\bar{\phi})}(0) = -2\sqrt{2k}p\bar{\mu}\epsilon^{-2\sqrt{2k}p}. \tag{2.54}$$

It is expected that the auxiliary fields play similar roles of cancelling the unwanted divergences in other correlators, though we will not analyze it in a systematic way.

2.4.3 Reflection relations

In previous subsection, we introduced the primary states $|j,m,s\rangle$ as highest weight states of $N=2$ superconformal algebra labelled by s. From a purely representation theoretical viewpoint, there are the following equivalence relations between them:

$$|j,m,s\rangle \sim |-j-1,m,s\rangle, \quad |j,\pm j,s\rangle \sim |\tilde{j},\mp\tilde{j},s\mp 1\rangle. \quad (\tilde{j}=-j-1-\tfrac{k}{2}) \tag{2.55}$$

Therefore, as in $N=0$ and 1 Liouville theories, we expect the following equivalence relations between operators

$$\begin{aligned} V^{j(s,\bar{s})}_{m,\bar{m}} &= R(j,m,\bar{m})V^{-j-1(s,\bar{s})}_{m,\bar{m}}, \\ V^{j(s,\bar{s})}_{\pm j,\pm j} &= R_\mp(j)\times V^{\tilde{j}(s\mp1,\bar{s}\mp1)}_{\mp\tilde{j},\mp\tilde{j}}, \quad \tilde{j}\equiv -j-\tfrac{k}{2}-1. \end{aligned} \tag{2.56}$$

We will refer to these relations as *reflection relations* and the coefficients $R, R_\mp$ as *reflection coefficients*. They should be independent of the labels s and $\bar{s}$, because they are coordinates along the S^1 corresponding to R-rotation which is an exact symmetry of the theory. $R(j,m,\bar{m})$ is easily obtained from the three-point structure constants $D_\pm$ and $F_\pm$:

$$R(j,m,\bar{m}) = -\nu^{2j+1}\frac{\Gamma(1+j+m)\Gamma(1+j-\bar{m})}{\Gamma(-j+m)\Gamma(-j-\bar{m})}\frac{\Gamma(-b^2(2j+1))\Gamma(-2j-1)}{\Gamma(b^2(2j+1))\Gamma(2j+1)}. \tag{2.57}$$

This can also be obtained from D_0 and F using the equality

$$\frac{F(j_a,m_a,\bar{m}_a)}{F(j_a,m_a,\bar{m}_a)|_{j_1\to -j_1-1}} = \frac{\Gamma(1+j_1+m_1)\Gamma(1+j_1-\bar{m}_1)}{\Gamma(-j_1+m_1)\Gamma(-j_1-\bar{m}_1)}\gamma(j_{2-3-1})\gamma(j_{3-1-2}). \tag{2.58}$$

$R_\pm(j)$ are obtained by using

$$F(j_a,m_a,\bar{m}_a)|_{m_1=\bar{m}_1=j_1} = \frac{(-)^{m_2-\bar{m}_2}\gamma(2j_1+1)\Gamma(1+j_2-m_2)\Gamma(1+j_3-m_3)}{\gamma(1+j_{1+2-3})\gamma(1+j_{1-2+3})\gamma(2+j_{1+2+3})\Gamma(\bar{m}_2-j_2)\Gamma(\bar{m}_3-j_3)}, \tag{2.59}$$

and taking the ratios of FD_0 and $F_\pm D_\pm$:

$$R_\pm(j) = \nu^{2j+1+\frac{k}{2}}\frac{\Gamma(-b^2(2j+1))}{\Gamma(-b^2(2\tilde{j}+1))}. \tag{2.60}$$

Finally, the two-point function for operators belonging to continuous representations can be written as

$$\begin{aligned} \langle V^{j_1(s_1\bar{s}_1)}_{m_1\bar{m}_1}(z_1)V^{j_2(s_2\bar{s}_2)}_{m_2\bar{m}_2}(z_2)\rangle &= z_{12}^{-2h_1}\bar{z}_{12}^{-2\bar{h}_1}\delta^2(m_1+m_2)\delta^2(s_1+s_2) \\ &\quad\times\left\{\delta\left(i(j_1+j_2+1)\right)+\delta\left(i(j_1-j_2)\right)R(j_1,m_1,\bar{m}_1)\right\}. \end{aligned} \tag{2.61}$$

2.5 OPE involving degenerate fields

The three-point structure constant and the reflection coefficients can also be obtained from the property of degenerate operators. The technology was first invented in bosonic Liouville theory in [4] (see also [7]).

We first study the OPEs involving degenerate operators. The operators $V^{1/2}_{M\bar{M}}$ and $V^{k/2}_{M\bar{M}}$ are the most important, because any other degenerate operators descend from their products. When multiplied on a generic operator $V^{j}_{m\bar{m}}$, they should satisfy the OPE formulae

$$\begin{aligned} V^{\frac{1}{2}}_{M\bar{M}}(z_1)V^{j}_{m\bar{m}}(z_2) &\sim e^{\frac{i\pi}{k}(M\bar{m}-\bar{M}m)}\sum_{\pm} z_{12}^{\frac{4Mm+1\mp(2j+1)}{2k}}\bar{z}_{12}^{\frac{4\bar{M}\bar{m}+1\mp(2j+1)}{2k}}C_{\pm}V^{j\pm\frac{1}{2}}_{m+M,\bar{m}+\bar{M}}(z_2), \\ V^{\frac{k}{2}}_{M\bar{M}}(z_1)V^{j}_{m\bar{m}}(z_2) &\sim e^{\frac{i\pi}{k}(M\bar{m}-\bar{M}m)}\sum_{\pm} z_{12}^{\frac{2Mm}{k}+\frac{1\mp(2j+1)}{2}}\bar{z}_{12}^{\frac{2\bar{M}\bar{m}}{k}+\frac{1\mp(2j+1)}{2}}\tilde{C}_{\pm}V^{j\pm\frac{k}{2}}_{m+\mu,\bar{m}+\bar{\mu}}(z_2) \\ &\quad +e^{\frac{i\pi}{k}(M\bar{m}-\bar{M}m)}\sum_{\updownarrow} z_{12}^{\frac{2Mm}{k}+\frac{k\hat{c}}{2}\mp(m+M)}\bar{z}_{12}^{\frac{2\bar{M}\bar{m}}{k}+\frac{k\hat{c}}{2}\mp(\bar{m}+\bar{M})}\tilde{C}_{\updownarrow}V^{j(\pm1,\pm1)}_{m+M\mp\frac{k\hat{c}}{2},\bar{m}+\bar{M}\mp\frac{k\hat{c}}{2}}(z_2). \end{aligned} \tag{2.62}$$

where the coefficients $C_{\pm}, \tilde{C}_{\pm,\updownarrow}$ are functions of $(M,\bar{M};j,m,\bar{m})$. Remember that $j=1/2$ operator must have $M,\bar{M}=\pm1/2$ in order to belong to a degenerate representation.

Throughout the paper we use the un-tilded or tilded letters (like C or $\tilde{C}$) for OPE coefficients involving $j=1/2$ or $j=k/2$ operators. The suffix $\pm$ indicates the channels in which j quantum number changes by $\pm1/2$ or $\pm k/2$, and $\updownarrow$ indicates the channels where the s quantum number changes by ±1.

The OPE coefficients can be calculated by the standard perturbative argument. The idea is that these finite number of terms in OPEs are the contributions from poles in three-point correlators, so are calculable as ordinary Wick contractions with some insertions of screening operators. We should use the two screening operators contained in the original action as perturbation terms

$$\mu S+\bar{\mu}\bar{S} = -\frac{k\mu}{\pi}\int d^2z e^{-\sqrt{\frac{k}{2}}\phi+iH} - \frac{k\bar{\mu}}{\pi}\int d^2z e^{-\sqrt{\frac{k}{2}}\bar{\phi}-iH}, \tag{2.63}$$

as well as the other (D-type) one,

$$\tilde{\mu}\tilde{S} = \frac{\tilde{\mu}}{4\pi}\int d^2z(\psi_+\psi_- - i\sqrt{2k}\partial\theta)(\bar{\psi}_+\bar{\psi}_- - i\sqrt{2k}\bar{\partial}\theta)e^{-\sqrt{\frac{2}{k}}\rho}. \tag{2.64}$$

The OPE coefficients are calculated as the ratios of three-point functions and two-point functions both of which are diverging, so that we only have to take the ratios of the residues. For example,

$$C_+(M\bar{M};jm\bar{m}) = e^{-\frac{i\pi}{k}(M\bar{m}-\bar{M}m)}\lim\frac{\langle V^{-j-3/2}_{-m-1/2,-\bar{m}+1/2}V^{1/2}_{1/2,-1/2}V^{j}_{m,\bar{m}}\rangle}{\langle V^{-j-3/2}_{-m-1/2,-\bar{m}+1/2}V^{j+1/2}_{m+1/2,\bar{m}-1/2}\rangle} = 1. \tag{2.65}$$

JHEP12(2006)061

In the same way, C_- is calculated as a Wick contraction with one screening operator $\tilde{\mu}\tilde{S}$ inserted:

$$\begin{aligned} C_-(M\bar{M};jm\bar{m}) &= e^{-\frac{i\pi}{k}(M\bar{m}-\bar{M}m)} \lim \frac{\langle(-\tilde{\mu}\tilde{S})V^{-j-1/2}_{-m-1/2,-\bar{m}+1/2}V^{1/2}_{1/2,-1/2}V^{j}_{m,\bar{m}}\rangle_{\text{free}}}{\langle V^{-j-1/2}_{-m-1/2,-\bar{m}+1/2}V^{j-1/2}_{m+1/2,\bar{m}-1/2}\rangle_{\text{free}}} \\ &= -\tfrac{\tilde{\mu}}{k^2}\gamma(-\tfrac{2j+1}{k})\gamma(\tfrac{2j}{k})\gamma(\tfrac{1}{k})(m-2jM)(\bar{m}-2j\bar{M}). \end{aligned} \tag{2.66}$$

The coefficients $\tilde{C}_{\pm,\updownarrow}$ are also calculated as Wick contractions with some $\mu S, \bar{\mu}\bar{S}$ inserted.

$$\begin{aligned} \tilde{C}_+(M\bar{M};jm\bar{m}) &= 1, \\ \tilde{C}_\uparrow(M\bar{M};jm\bar{m}) &= k\mu e^{\frac{i\pi}{2}(\bar{m}-m+M-\bar{M})}\frac{\Gamma(1+j-m)\Gamma(1+\frac{k}{2}-M)\Gamma(-j-\frac{k}{2}+\bar{m}+\bar{M}-1)}{\Gamma(-j+\bar{m})\Gamma(-\frac{k}{2}+\bar{M})\Gamma(2+j+\frac{k}{2}-m-M)}, \\ \tilde{C}_\downarrow(M\bar{M};jm\bar{m}) &= k\bar{\mu}e^{-\frac{i\pi}{2}(\bar{m}-m+M-\bar{M})}\frac{\Gamma(1+j+m)\Gamma(1+\frac{k}{2}+M)\Gamma(-j-\frac{k}{2}-\bar{m}-\bar{M}-1)}{\Gamma(-j-\bar{m})\Gamma(-\frac{k}{2}-\bar{M})\Gamma(2+j+\frac{k}{2}+m+M)}, \\ \tilde{C}_-(M\bar{M};jm\bar{m}) &= k^2\mu\bar{\mu}\gamma(-2j-1)\gamma(1+2j-k)\frac{\Gamma(1+j+m)\Gamma(1+j-\bar{m})}{\Gamma(-j+m)\Gamma(-j-\bar{m})} \\ &\quad\times\frac{\Gamma(-j+\frac{k}{2}+m+M)\Gamma(-j+\frac{k}{2}-\bar{m}-\bar{M})}{\Gamma(1+j-\frac{k}{2}+m+M)\Gamma(1+j-\frac{k}{2}-\bar{m}-\bar{M})}. \end{aligned} \tag{2.67}$$

Here we restricted to operators which are not spectral flowed, but the OPE coefficients depend on the labels $s, \bar{s}$ at most through cocycle factors. These coefficients are all obtained by the repeated use of the formula

$$\int d^2z z^{\alpha}\bar{z}^{\bar{\alpha}}(1-z)^{\beta}(1-\bar{z})^{\bar{\beta}} = \pi\frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(-\bar{\alpha}-\bar{\beta}-1)}{\Gamma(-\bar{\alpha})\Gamma(-\bar{\beta})\Gamma(\alpha+\beta+2)}. \tag{2.68}$$

Setting $\mu = \bar{\mu} = \nu^{\frac{k}{2}}$ and combining the OPE formulae with the reflection symmetry we find that the reflection coefficients are given by

$$\begin{aligned} R(j,m,\bar{m}) &= -\nu^{2j+1}\frac{\Gamma(1+j-m)\Gamma(1+j+\bar{m})}{\Gamma(-j-m)\Gamma(-j+\bar{m})}\frac{\Gamma(-2j-1)\Gamma(-\frac{2j+1}{k})}{\Gamma(2j+1)\Gamma(\frac{2j+1}{k})}, \\ R_{\pm}(j) &= \nu^{2j+1+\frac{k}{2}}\gamma(-\tfrac{2j+1}{k}). \end{aligned} \tag{2.69}$$

consistently with the expression obtained from three-point structure constants. At the same time, we also find the relation between coupling constants ν and $\tilde{\mu}$:

$$\nu = \tilde{\mu}\gamma(\tfrac{1}{k}). \tag{2.70}$$

3. N=2 Liouville theory with boundary

Now we turn to the analysis of the theory in the presence of boundary. As boundary conditions we only consider those preserving a half of superconformal symmetry of the theory without boundary. The simplest worldsheet with boundary is the upper half-plane or the disc, through the analysis of which one can classify all the possible boundary states.

Following the recent works on the boundary Liouville [9] and $N = 1$ super-Liouville theories [12, 13], we first analyze the annulus amplitudes using the modular transformation

JHEP12(2006)061

property of characters of $N=2$ superconformal algebra. They have been studied in some recent works [14, 16, 17] and [18, 19], but let us analyze them carefully, taking the proper account of the quantization law of θ-momentum/winding number.

On the theory on the upper half-plane, there are two classes of boundary conditions on the real line:

$$\begin{aligned}
&\textbf{A-type} : T(z)=\bar{T}(\bar{z}), \quad T_F^{\pm}(z)=e^{\pm 2\pi i\alpha}\bar{T}_F^{\mp}(\bar{z}), \quad J(z)=-\bar{J}(\bar{z}), \\
&\textbf{B-type} : T(z)=\bar{T}(\bar{z}), \quad T_F^{\pm}(z)=e^{\pm 2\pi i\alpha}\bar{T}_F^{\pm}(\bar{z}), \quad J(z)=\bar{J}(\bar{z}).
\end{aligned} \tag{3.1}$$

where α denotes the angle of R-rotation by $J_0 \pm \bar{J}_0$. Both of them preserve a copy of $N=2$ superconformal algebra. By a conformal map that transforms the upper half-plane to the unit disc, they are transformed to the condition on *boundary states*. For A-type boundary states it becomes,

$$\begin{aligned}
0 &= \langle A^\alpha|(L_n-\bar{L}_{-n}), & (L_n-\bar{L}_{-n})|A^\alpha\rangle &= 0, \\
0 &= \langle A^\alpha|(G_r^{\pm}+ie^{\pm 2\pi i\alpha}\bar{G}_{-r}^{\mp}), & (G_r^{\pm}-ie^{\pm 2\pi i\alpha}\bar{G}_{-r}^{\mp})|A^\alpha\rangle &= 0, \\
0 &= \langle A^\alpha|(J_n-\bar{J}_{-n}), & (J_n-\bar{J}_{-n})|A^\alpha\rangle &= 0,
\end{aligned} \tag{3.2}$$

while the condition on B-types is

$$\begin{aligned}
0 &= \langle B^\alpha|(L_n-\bar{L}_{-n}), & (L_n-\bar{L}_{-n})|B^\alpha\rangle &= 0, \\
0 &= \langle B^\alpha|(G_r^{\pm}+ie^{\pm 2\pi i\alpha}\bar{G}_{-r}^{\pm}), & (G_r^{\pm}-ie^{\pm 2\pi i\alpha}\bar{G}_{-r}^{\pm})|B^\alpha\rangle &= 0, \\
0 &= \langle B^\alpha|(J_n+\bar{J}_{-n}), & (J_n+\bar{J}_{-n})|B^\alpha\rangle &= 0.
\end{aligned} \tag{3.3}$$

D-branes are described as boundary states, or the solutions to (3.2) or (3.3) supporting a well-defined spectrum of open string states. Ishibashi states form the basis of solutions to the boundary condition, and are constructed by summing up all the descendants of a single primary state. We define A-type Ishibashi states by (here we use $h_0=\frac{m^2-j(j+1)}{k}, q_0=\frac{2m}{k}$)

$$\begin{aligned}
\langle\langle A^\alpha_{j,m,\beta}| &= \\
& e^{-2\pi i\alpha(\frac{2m}{k}+\beta\hat{c})}\langle V^{j(\beta,\beta)}_{m,m}|\left(1+\frac{ie^{-2\pi i\alpha}}{2h_0-q_0}G^+_{\frac{1}{2}+\beta}\bar{G}^+_{\frac{1}{2}+\beta}+\frac{ie^{2\pi i\alpha}}{2h_0+q_0}G^-_{\frac{1}{2}-\beta}\bar{G}^-_{\frac{1}{2}-\beta}+\cdots\right), \\
|A^\alpha_{j,m,\beta}\rangle\rangle &= \\
& \left(\cdots+\frac{ie^{-2\pi i\alpha}}{2h_0-q_0}G^+_{-\frac{1}{2}-\beta}\bar{G}^+_{-\frac{1}{2}-\beta}+\frac{ie^{2\pi i\alpha}}{2h_0+q_0}G^-_{-\frac{1}{2}+\beta}\bar{G}^-_{-\frac{1}{2}+\beta}+1\right)|V^{j(\beta,\beta)}_{m,m}\rangle e^{-2\pi i\alpha(\frac{2m}{k}+\beta\hat{c})}, \\
& (m+\beta\in\tfrac{k}{2}\mathbb{Z})
\end{aligned} \tag{3.4}$$

and B-type Ishibashi states by

$$\begin{aligned}
\langle\langle B^\alpha_{j,m,\beta}| &= \\
& e^{-2\pi i\alpha(\frac{2m}{k}+\beta\hat{c})}\langle V^{j(\beta,-\beta)}_{m,-m}|\left(1+\frac{ie^{-2\pi i\alpha}}{2h_0-q_0}G^+_{\frac{1}{2}+\beta}\bar{G}^-_{\frac{1}{2}+\beta}+\frac{ie^{2\pi i\alpha}}{2h_0+q_0}G^-_{\frac{1}{2}-\beta}\bar{G}^+_{\frac{1}{2}-\beta}+\cdots\right), \\
|B^\alpha_{j,m,\beta}\rangle\rangle &= \\
& \left(\cdots+\frac{ie^{-2\pi i\alpha}}{2h_0-q_0}G^+_{-\frac{1}{2}-\beta}\bar{G}^-_{-\frac{1}{2}-\beta}+\frac{ie^{2\pi i\alpha}}{2h_0+q_0}G^-_{-\frac{1}{2}+\beta}\bar{G}^+_{-\frac{1}{2}+\beta}+1\right)|V^{j(\beta,-\beta)}_{m,-m}\rangle e^{-2\pi i\alpha(\frac{2m}{k}+\beta\hat{c})}, \\
& (m\in\tfrac{1}{2}\mathbb{Z})
\end{aligned} \tag{3.5}$$

Note the restriction on m arising from the θ-momentum/winding number quantization law. Note that, since $V^{j(\beta,\beta)}_{m,m}$ can be transformed to $V^{j(\beta+n,\beta+n)}_{m-n,m-n}$ for any integer n by a multiplication of supercharges, there are proportionality relations between Ishibashi states

$$\langle\langle A^{\alpha}_{j,m,\beta}| \sim \langle\langle A^{\alpha}_{j,m+n,\beta-n}|, \quad |A^{\alpha}_{j,m,\beta}\rangle\rangle \sim |A^{\alpha}_{j,m+n,\beta-n}\rangle\rangle. \tag{3.6}$$

The same holds also for B-type Ishibashi states. In this paper we only consider the Ishibashi states lying in continuous representations ($j \in -\frac{1}{2} + i\mathbb{R}$), and set their normalization by the formula

$$\begin{aligned}
&\langle\langle A^{\alpha}_{j,m,\beta}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|A^{\alpha'}_{j',m',-\beta}\rangle\rangle \\
&\quad = 2\pi\delta_{m+m',0}\left\{\delta(i(j+j'+1)) + \delta(i(j-j'))R(j,m,m)\right\}\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha), \\
&\langle\langle B^{\alpha}_{j,m,\beta}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|B^{\alpha'}_{j',m',-\beta}\rangle\rangle \\
&\quad = 2\pi\delta_{m+m',0}\left\{\delta(i(j+j'+1)) + \delta(i(j-j'))R(j,m,-m)\right\}\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha),
\end{aligned} \tag{3.7}$$

where $R(j,m,\bar{m})$ is the reflection coefficient for bulk operators, and $\chi_{j,m,\beta}(\tau,\alpha)$ is the $N=2$ character for continuous representation

$$\chi_{j,m,\beta}(\tau,\alpha) \equiv q^{\frac{m^2}{k}-\frac{(2j+1)^2}{4k}+\frac{\beta^2}{2}} z^{\frac{2m}{k}+\beta}\vartheta(\alpha+\beta\tau,\tau)\eta(\tau)^{-3}, \tag{3.8}$$

with $q = e^{2\pi i\tau}$, $z = e^{2\pi i\alpha}$. $\theta(\nu,\tau)$ is Jacobi theta function and $\eta(\tau)$ is Dedekind eta function; see the appendix for their definition and modular transformation property. It follows from this that the character is periodic in β with period 1, corresponding to the equivalence of Ishibashi states (3.6).

D-branes are expressed as suitable superpositions of the Ishibashi states. We call them as A-branes or B-branes, depending on the choice of boundary conditions. A-branes are point-like along θ-direction in the sense that they source closed string states without winding number along θ-direction. Similarly, B-branes are winding around θ-direction.

We will also consider the spectrum of open string states or corresponding boundary operators between two arbitrary D-branes. We will restrict our discussion mainly to those between the same type of branes. We will later consider the boundary primary operators labelled by (l,m,s), and denote them as $[B^{l(s)}_m]^X_{X'}$ making explicit the dependence on two D-branes X and X' they are ending on. Similarly to the bulk operators, the boundary operators also obey certain quantization law of momentum or winding number along θ-direction.

Let us first consider the open string states between two A-branes. Since A-branes are not wrapping around S^1, they do not carry θ-momentum. Therefore the open string states with both ends on the same A-brane only have quantized winding numbers, $m \in \mathbb{Z}$. (Later we will see a mild modification to this.) For a generic pair of A-branes it will be shifted as $m \in \mathbb{Z} + \delta$, but it should still be integer-spaced. The index s should also be quantized. Generic A-brane satisfy the boundary condition twisted by α, and the open strings stretched between two A-branes labelled by α and α' are in the $(\alpha-\alpha')$-th spectral flowed sector. This can be understood in the following way. If we put a boundary operator at the origin and A-branes with labels (α, α') on the negative and positive real axis, then we obtain

$$T_F^{\pm}(ze^{2\pi i}) = e^{\pm 2\pi i(\alpha-\alpha')} T_F^{\pm}(z), \tag{3.9}$$

indicating that the boundary operator should belong to the spectral flowed sector. One can argue in a similar way for B-branes, so the quantization laws are summarized as follows:

$$[B_m^{l(s)}]_{X'}^{X} : \begin{cases} X, X' \ \textbf{A-branes} \ \Rightarrow m \in \mathbb{Z}+\delta, \qquad\quad s \in \mathbb{Z}+\alpha-\alpha', \\ X, X' \ \textbf{B-branes} \ \Rightarrow m+s \in k\mathbb{Z}+\delta, \ s \in \mathbb{Z}+\alpha-\alpha'. \end{cases} \tag{3.10}$$

One can immediately check the compatibility with the superconformal symmetry: if a primary operator connects two D-branes, so does any of its descendants.

3.1 Modular bootstrap for A-branes

From the previous discussion we expect that the open string spectrum between two A-branes involves summing over m quantum number with unit periodicity. Based on this, we propose that

the open string spectrum between A-branes is a sum over integer spectral flow.

This means that the presence of $[B_m^{l(s)}]_{X'}^{X}$ implies the presence of $[B_m^{l(s+n)}]_{X'}^{X}$ for any integer n. This is proved in the following way. Consider an open string state between two A-branes labelled by α, α'. Its s label has to satisfy $s+\alpha'-\alpha \in \mathbb{Z}$. Now rotate one A-brane adiabatically so that α increases by one. The A-brane labelled by α should come back to itself up to an overall phase, so the open string spectrum in particular will not change by the unit shift of α. On the other hand, during the adiabatic process the s label of each open string state increases, and is shifted by one in the end. This means the invariance of open string spectrum under integer spectral flow. So the open string amplitudes between A-branes involve characters of a *large* $N = 2$ superconformal algebra which includes integer spectral flows. Note that the sum of characters over integer spectral flows is equivalent to the sum over integer shifts of m quantum number, due to the periodicity of the character $\chi_{j,m,\beta}(\tau, \alpha)$ explained before.

Let us start with presenting several useful formulae for later calculations. First, we will frequently consider the sum of characters spectral flowed by integer amounts. So let us work out the modular transformation law for such quantity here. Using the Gauss integral and Poisson resummation formula one finds

$$\begin{aligned} &e^{-2\pi i\alpha\beta} \sum_{n\in\mathbb{Z}+\alpha} e^{-\frac{4\pi i(M+n)\beta}{k}} \chi_{J,M+n,n}(\tau_o, \beta) \\ &= -i \sum_{m\in\frac{k}{2}\mathbb{Z}} \int_{\mathcal{C}_0} dj e^{-\frac{4\pi i M m}{k} + \frac{i\pi}{k}(2j+1)(2J+1)} \chi_{j,m,\beta}(\tau_c, -\alpha). \qquad (\mathcal{C}_0 \equiv \{-\tfrac{1}{2} + i\mathbb{R}\}) \end{aligned} \tag{3.11}$$

Next, let us present a formula involving characters for chiral representations.

$$\frac{\chi_{J,M,\alpha}(\tau,\beta)}{1+e^{2\pi i\beta} q^{M\pm(J+\frac{1}{2})}} = \sum_{l\in\mathbb{Z}_{\geq 0}} (-)^l \chi_{J\mp\frac{kl}{2}, M+\frac{kl}{2}, \alpha}(\tau, \beta),$$

$$\frac{\chi_{J,M,\alpha}(\tau,\beta)}{1+e^{-2\pi i\beta}q^{-M\pm(J+\frac{1}{2})}} = \sum_{l\in\mathbb{Z}_{\geq 0}} (-)^l \chi_{J\mp\frac{kl}{2},M-\frac{kl}{2},\alpha}(\tau,\beta). \qquad (3.12)$$

3.1.1 Identity representation

Of all the boundary states satisfying the A-type boundary condition, the most important is the one corresponding to the identity representation, $\langle A_{[1]}|$ and $|A_{[1]}\rangle$. We start from the fact that the annulus amplitude with both ends on it is given by the sum of the character for identity representation over integer spectral flows.

$$\begin{aligned} Z &= \langle A^{\alpha,\beta}_{[1]}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|A^{\alpha',-\beta}_{[1]}\rangle \\ &= \sum_{n\in\mathbb{Z}+\alpha-\alpha'} y^{\alpha'-\alpha-\frac{2n}{k}} \frac{\chi_{0,n,n}(\tau_o,\beta)(1-q_o)}{(1+yq_o^{\frac{1}{2}+n})(1+y^{-1}q_o^{\frac{1}{2}-n})} \qquad (y=e^{2\pi i\beta}) \\ &= \sum_{l\in\mathbb{Z}_{\geq 0}}(-)^l \sum_{n\in\mathbb{Z}+\alpha-\alpha'} y^{\alpha'-\alpha-\frac{2n}{k}}\left\{\chi_{-\frac{kl}{2},n+\frac{kl}{2},n}(\tau_o,\beta) - \chi_{\frac{kl}{2},n+\frac{kl}{2},n}(\tau_o,\beta)\right\}. \qquad (3.13)\end{aligned}$$

The open strings are in the $(\alpha-\alpha')$-th spectral flowed sector. In the second line, the powers of y in the sum is chosen in accordance with the periodicity of the label $\alpha\sim\alpha+1$ of boundary states. It therefore follows that, when the closed string states are chosen from (β,β)-spectral flowed sector, the trace over open string states should be taken with the phase $e^{2\pi i\beta F}$, where F is defined by

$$F[B_0^{0(n)}] \equiv n-\alpha+\alpha' = J_0[B_0^{0(n)}]+\alpha'-\alpha-\tfrac{2n}{k}, \qquad F[G_r^{\pm}]=\pm 1. \qquad (3.14)$$

After the modular S transformation, the annulus amplitude is expressed as a sum over closed string exchanges,

$$\begin{aligned} Z &= -i\sum_{l\in\mathbb{Z}_{\geq 0}}(-)^l \int_{\mathcal{C}_0} dj \sum_{m+\beta\in\frac{k}{2}\mathbb{Z}} e^{\frac{i\pi}{k}(2j+1)} \times \\ &\qquad \left\{e^{-i\pi l(2m+2j+1)} - e^{-i\pi l(2m-2j-1)}\right\}\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha) \\ &\simeq \int_{\mathcal{C}_0} dj \sum_{m+\beta\in\frac{k}{2}\mathbb{Z}} \frac{i\sin\pi(2j+1)\sin\frac{\pi}{k}(2j+1)}{2\sin(j+m)\pi\sin(j-m)\pi}\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha), \qquad (3.15)\end{aligned}$$

where $\simeq$ means the equality up to possible emergence of discrete series states from changing the order of l-sum and j-integration. On the other hand, A-branes are written as superpositions of A-type Ishibashi states

$$\begin{aligned} \langle A^{\alpha,\beta}_{[1]}| &= \sum_{m+\beta\in\frac{kZ}{2}} \int_{\mathcal{C}_0}\frac{dj}{2\pi i} U_{[1]}(-j-1,-m,-\beta) \times \langle\langle A^{\alpha}_{j,m,\beta}| + \text{(discrete reps.)}, \\ |A^{\alpha,\beta}_{[1]}\rangle &= \sum_{m+\beta\in\frac{kZ}{2}} \int_{\mathcal{C}_0}\frac{dj}{2\pi i}|A^{\alpha}_{j,m,\beta}\rangle\rangle \times U_{[1]}(-j-1,-m,-\beta) + \text{(discrete reps.)}, \qquad (3.16)\end{aligned}$$

the wave function for the identity A-brane $U_{[1]}(j,m,\beta)$ has to satisfy

$$\begin{aligned} U_{[1]}(j,m,\beta) &= R(j,m,m)U_{[1]}(-j-1,m,\beta), \\ U_{[1]}(j,m,\beta)U_{[1]}(-j-1,-m,-\beta) &= -\frac{\pi\sin\pi(2j+1)\sin\frac{\pi(2j+1)}{k}}{2\sin\pi(j+m)\sin\pi(j-m)}. \qquad (3.17)\end{aligned}$$

So we obtain, up to ± sign,

$$U_{[1]}(j,m,\beta) \;=\; (\tfrac{k\pi}{2})^{\frac{1}{2}}\nu^{j+\frac{1}{2}}\frac{\Gamma(1+j+m)\Gamma(1+j-m)}{\Gamma(2j+2)\Gamma(\frac{2j+1}{k})}. \tag{3.18}$$

The wave functions for other A-branes are obtained by considering annulus amplitudes bounded by one identity and one generic A-branes. In the following we consider five classes of them, and we label them by the highest weights $|J,M\rangle$ of $N=2$ superconformal algebra. In the following we will simply neglect the contribution from closed string states in discrete representations, because the wave function $U(j,m,\beta)$ for $j\in -\frac{1}{2}+i\mathbb{R}$ is enough to determine the disc one-point function of bulk operators completely under the assumption of analyticity.

3.1.2 Non-chiral non-degenerate representations

The first example we consider is the A-brane $|A_{[J,M]}\rangle$ corresponding to the Verma module over highest weight state $|J,M\rangle$. The annulus amplitude between this and an identity A-branes is given by a sum of characters over integer spectral flows,

$$\begin{aligned}\langle A^{\alpha,\beta}_{[1]}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|A^{\alpha',-\beta}_{[J,M]}\rangle \;&=\; \sum_{n\in\mathbb{Z}+\alpha-\alpha'} y^{\alpha'-\alpha-\frac{2}{k}(M+n)}\chi_{J,M+n,n}(\tau_o,\beta)\\ = -i\sum_{m+\beta\in\frac{k}{2}\mathbb{Z}}&\int_{\mathcal{C}_0} dj\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha)e^{-\frac{4\pi iM}{k}(m+\beta)}\cos\{\tfrac{\pi}{k}(2j+1)(2J+1)\}.\end{aligned} \tag{3.19}$$

From this we obtain

$$U_{[1]}(j,m,\beta)U_{[J,M]}(-j-1,-m,-\beta)=\pi e^{-\frac{4\pi iM}{k}(m+\beta)}\cos\{\tfrac{\pi}{k}(2j+1)(2J+1)\}. \tag{3.20}$$

The wave function for this A-brane thus becomes

$$U_{[J,M]}(j,m,\beta)=(\tfrac{2\pi}{k})^{\frac{1}{2}}\nu^{j+\frac{1}{2}}\frac{\Gamma(-2j)\Gamma(-\frac{2j+1}{k})}{\Gamma(-j+m)\Gamma(-j-m)}e^{\frac{4\pi iM}{k}(m+\beta)}\cos\{\tfrac{\pi}{k}(2j+1)(2J+1)\}. \tag{3.21}$$

3.1.3 Non-chiral degenerate representations

When $J=J_{r,s}=\frac{1}{2}(r-1+ks)$ $(r,s\in\mathbb{Z}_{>0})$ the Verma module over $|J,M\rangle$ has a null vector at the level rs, and an irreducible representation is defined by the subtraction of the null submodule. Denoting the corresponding A-brane by $|A_{[J_{r,s},M]}\rangle$, the annulus amplitude between this and the identity A-branes becomes

$$\begin{aligned}&\langle A^{\alpha,\beta}_{[1]}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|A^{\alpha',-\beta}_{[J_{r,s},M]}\rangle\\ &=\sum_{n\in\mathbb{Z}+\alpha-\alpha'} y^{\alpha'-\alpha-\frac{2}{k}(M+n)}\left\{\chi_{J_{r,s},M+n,n}(\tau_o,\beta)-\chi_{J_{-r,s},M+n,n}(\tau_o,\beta)\right\},\end{aligned} \tag{3.22}$$

from which we obtain, in the same way as before,

$$U_{[J_{r,s},M]}(j,m,\beta)=-2(\tfrac{2\pi}{k})^{\frac{1}{2}}\nu^{j+\frac{1}{2}}\frac{\Gamma(-2j)\Gamma(-\frac{2j+1}{k})}{\Gamma(-j+m)\Gamma(-j-m)}e^{\frac{4\pi iM}{k}(m+\beta)}\sin\frac{(2j+1)r\pi}{k}\sin(2j+1)s\pi. \tag{3.23}$$

From the momentum quantization for bulk operators $m+\beta\in\frac{k}{2}\mathbb{Z}$ it follows that the label M has period 1 for these two classes of branes.

3.1.4 Anti-chiral representations

When the highest weight $|J, M\rangle$ satisfies $J - M \in \mathbb{Z}_{\geq 0}$, then an irreducible representation is obtained by putting

$$0 = G^-_{-J+M-\frac{1}{2}} \cdots G^-_{-\frac{3}{2}} G^-_{-\frac{1}{2}} |J, M\rangle. \tag{3.24}$$

The case $J = M$ gives an anti-chiral representation, and other cases are its spectral flow. Denoting the corresponding A-branes by $|A_{[J,M]^-}\rangle$, the annulus amplitude between this and the identity A-branes is calculated as follows:

$$\langle A^{\alpha,\beta}_{[1]} | e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})} | A^{\alpha',-\beta}_{[J,M]^-}\rangle = \sum_{n\in\mathbb{Z}+\alpha-\alpha'} y^{\alpha'-\alpha-\frac{2}{k}(M+n)} \frac{\chi_{J,M+n,n}(\tau_o,\beta)}{1+y^{-1}q_o^{\frac{1}{2}+J-M-n}}. \tag{3.25}$$

From this we obtain

$$\begin{aligned} U_{[J,M]^-}(j,m,\beta) &= i(8k\pi)^{-\frac{1}{2}} \nu^{j+\frac{1}{2}} e^{\frac{4\pi i M}{k}(m+\beta)} \Gamma(-2j)\Gamma(-\tfrac{2j+1}{k}) \\ &\times \left\{ e^{i\pi(m+j)+\frac{i\pi}{k}(2j+1)(2J+1)} \frac{\Gamma(1+j+m)}{\Gamma(-j+m)} - e^{i\pi(m-j)-\frac{i\pi}{k}(2j+1)(2J+1)} \frac{\Gamma(1+j-m)}{\Gamma(-j-m)} \right\}. \end{aligned} \tag{3.26}$$

3.1.5 Chiral representations

Similarly to the above, when the highest weight satisfies $J + M \in \mathbb{Z}_{\geq 0}$ the irreducible representations are defined by the null vector equation

$$0 = G^+_{-J-M-\frac{1}{2}} \cdots G^+_{-\frac{3}{2}} G^+_{-\frac{1}{2}} |J, M\rangle. \tag{3.27}$$

They are chiral representations or their spectral flows. The corresponding A-branes are denoted as $|A_{[J,M]^+}\rangle$, and from the analysis of annulus amplitude we obtain

$$\begin{aligned} U_{[J,M]^+}(j,m,\beta) &= i(8k\pi)^{-\frac{1}{2}} \nu^{j+\frac{1}{2}} e^{\frac{4\pi i M}{k}(m+\beta)} \Gamma(-2j)\Gamma(-\tfrac{2j+1}{k}) \\ &\times \left\{ e^{i\pi(-m+j)+\frac{i\pi}{k}(2j+1)(2J+1)} \frac{\Gamma(1+j-m)}{\Gamma(-j-m)} - e^{i\pi(-m-j)-\frac{i\pi}{k}(2j+1)(2J+1)} \frac{\Gamma(1+j+m)}{\Gamma(-j+m)} \right\}. \end{aligned} \tag{3.28}$$

Taking the quantization conditions on M and $m + \beta$ into account, one finds that the wave functions for (anti-)chiral A-branes are independent of M. We can also check the following equivalence as required from representation theory:

$$|A_{[J,\pm J]^\mp}\rangle = \text{const.} \times |A_{[\tilde{J},\mp\tilde{J}]^\pm}\rangle \quad (\tilde{J} = -J - 1 - \tfrac{k}{2}). \tag{3.29}$$

3.1.6 Degenerate chiral representations

When $J \pm M$ are both nonnegative integers, the Verma module has two independent null vectors defined by (3.24) and (3.27). By setting them to zero we obtain an irreducible representation which we call as degenerate chiral. The corresponding A-branes will be denoted as $|A_{[J,M]^{\rm dc}}\rangle$. The calculation of annulus amplitudes involving them is a little complicated. By a little thought one finds that the representation space is spanned by the following vectors

$$
\begin{aligned}
&\bigoplus_{p=1}^{J+M}\left\{\text{polynomial of } L_{n\le -2},\ G^{\pm}_{r\le\mp p-3/2},\ J_{n\le -1}\right\} G^{+}_{-p+\frac{1}{2}}\cdots G^{+}_{-\frac{3}{2}}G^{+}_{-\frac{1}{2}}|J,M\rangle \\
&\quad\oplus\left\{\text{polynomial of } L_{n\le -2},\ G^{\pm}_{r\le -3/2},\ J_{n\le -1}\right\}|J,M\rangle \\
&\oplus\bigoplus_{p=1}^{J-M}\left\{\text{polynomial of } L_{n\le -2},\ G^{\pm}_{r\le\pm p-3/2},\ J_{n\le -1}\right\} G^{-}_{-p+\frac{1}{2}}\cdots G^{-}_{-\frac{3}{2}}G^{-}_{-\frac{1}{2}}|J,M\rangle .
\end{aligned}
\tag{3.30}
$$

So the character for this representation spectral flowed by α units is given by

$$
\begin{aligned}
\mathrm{Tr}[y^F q^{L_0-\frac{c}{24}}] &= \sum_{p=M-J}^{M+J} q^{\frac{(M+\alpha)^2-J(J+1)}{k}+\frac{(p+\alpha)^2}{2}-\frac{c}{24}} y^p \prod_{n\ge 1}\frac{(1+yq^{n+\frac{1}{2}+p+\alpha})(1+y^{-1}q^{n+\frac{1}{2}-p-\alpha})}{(1-q^n)(1-q^{n+1})} \\
&= \sum_{p=M-J}^{M+J}\frac{y^{-\frac{2M}{k}-\alpha\hat{c}}\chi_{J,M+\alpha,p+\alpha}(\tau,\beta)(1-q)}{(1+yq^{\frac{1}{2}+p+\alpha})(1+y^{-1}q^{\frac{1}{2}-p-\alpha})} .
\end{aligned}
\tag{3.31}
$$

After summing over spectral flow we obtain the annulus amplitude bounded by $|A_{[J,M]^{\mathrm{dc}}}\rangle$ and the identity A-branes:

$$
\begin{aligned}
Z &= \langle A^{\alpha,\beta}_{[1]}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|A^{\alpha',-\beta}_{[J,M]^{\mathrm{dc}}}\rangle \\
&= \sum_{n\in\mathbb{Z}+\alpha-\alpha'}\sum_{p=M-J}^{M+J} y^{\alpha'-\alpha-\frac{2}{k}(M+n)}\frac{\chi_{J,M+n,n+p}(\tau,\beta)(1-q_o)}{(1+yq_o^{\frac{1}{2}+p+n})(1+y^{-1}q_o^{\frac{1}{2}-p-n})} .
\end{aligned}
\tag{3.32}
$$

The wave function thus becomes

$$
U_{[J,M]^{\mathrm{dc}}}(j,m,\beta)=e^{\frac{4\pi i M}{k}(m+\beta)}\left(\frac{k\pi}{2}\right)^{\frac{1}{2}}\nu^{j+\frac{1}{2}}\frac{\Gamma(1+j-m)\Gamma(1+j+m)\sin\{\frac{\pi}{k}(2J+1)(2j+1)\}}{\Gamma(2j+2)\Gamma(\frac{2j+1}{k})\sin\frac{(2j+1)\pi}{k}} .
\tag{3.33}
$$

These A-branes $|A_{[J,M]^{\mathrm{dc}}}\rangle$ are also independent of M, and labelled by a single positive integer $n=2J+1$. So we also denote them by $|A_{[n]}\rangle$. The case $J=M=0$ corresponds to the identity A-brane $|A_{[1]}\rangle$ analyzed previously.

3.2 Modular bootstrap for B-branes

One might expect that the wave functions for B-branes are obtained through a similar analysis of annulus amplitudes, but it turns out not the case.

Based on the free field picture, we found that the boundary operator $[B^{l(s)}_m]$ between B-branes satisfy the momentum quantization law $m+s\in k\mathbb{Z}+\mathrm{const}$. So the annulus amplitudes as seen from the open string channel should be sums over characters labelled by (l,m,s) with the above constraint. However, the simple shift of m by $k\mathbb{Z}$ is not an isomorphism of representations of $N=2$ superconformal algebra, especially for irrational k and (l,m,s) belonging to chiral representations. The simple mixtures of chiral and non-chiral representations will not lead to annulus amplitudes with nice modular transformation property, i.e. they will not have a sensible closed string channel interpretation. In particular, for irrational k, it is difficult to think of spectrum of open string states between identity B-branes. If there is no identity brane, then the modular bootstrap analysis for B-branes will not be as powerful as it was for A-branes.

For A-branes, the periodicity under R-rotation $\alpha \to \alpha + 1$ was the key in finding the correct open string spectrum. However, for B-branes this does not seem to yield any useful information on which representations to sum over. Consider an open string state labelled by (l, m, s) and stretched between two B-branes, and what happens to it when one of the B-branes is R-rotated once in an adiabatic way so that it returns to itself. s will increase by one as before, but this time m will decrease by one as well in order to meet with the momentum quantization law. The new state is related to the original state by the action of supercurrent (2.23), so the two states are within the same representation space of boundary superconformal algebra.

When k is an integer, there is a candidate for open string spectrum between identity B-branes, because then the sum over $k\mathbb{Z}$ shifts of the quantum number $m + s$ can be interpreted as the sum over $k\mathbb{Z}$ spectral flows. This is expressed in terms of characters as follows:

$$\begin{aligned}
&\sum_{m\in\frac{1}{2}\mathbb{Z}+\beta} \int_{\mathcal{C}_0} \frac{dj}{ik} e^{\frac{i\pi}{k}(2j+1)(2J+1)-\frac{4\pi i M m}{k}} \chi_{j,m,\beta}(\tau_c, -\alpha) \\
&= \sum_{n\in k\mathbb{Z}} e^{-2\pi i\hat{c}\alpha\beta-\frac{4\pi i\beta M}{k}} \chi_{J,M+n+\alpha,\alpha}(\tau_o, \beta) \\
&\overset{k\in\mathbb{Z}}{=} \sum_{n\in k\mathbb{Z}+\alpha} e^{-2\pi i\hat{c}\alpha\beta-\frac{4\pi i\beta M}{k}} \chi_{J,M+n,n}(\tau_o, \beta).
\end{aligned} \tag{3.34}$$

The sum over $k\mathbb{Z}$ spectral flows of identity character has a nice modular transformation property. Although this might be extended to the cases with rational k by a suitable orbifolding, we will continue to focus on integer k.

Let us go on and see whether we can re-write the annulus amplitude and obtain an analytic expression for wave function in consistency with the reflection relation of bulk operators. Denoting by $T_{[1]}(j, m, \beta)$ the wave function for the identity B-brane, one finds

$$\begin{aligned}
T_{[1]}(j, m, \beta) &= R(j, m, -m) T_{[1]}(-j-1, m, \beta), \\
T_{[1]}(j, m, \beta) T_{[1]}(-j-1, -m, -\beta) &= -\frac{\pi \sin \pi(2j+1) \sin \frac{\pi(2j+1)}{k}}{2k \sin \pi(j+m) \sin \pi(j-m)}.
\end{aligned} \tag{3.35}$$

Note that $R(j, m, -m) = e^{2\pi i m} R(j, m, m)$ under the quantization condition $m \in \frac{1}{2}\mathbb{Z}$. The first equation is solved by

$$T_{[1]}(j, m, \beta) = (\tfrac{\pi}{2})\nu^{j+\frac{1}{2}} \frac{\Gamma(1+j+m)\Gamma(1+j-m)}{\Gamma(2j+2)\Gamma(\frac{2j+1}{k})} \left\{ \hat{T}(j) + e^{2\pi i m} \hat{T}(-j-1) \right\} \tag{3.36}$$

and the second one yields

$$2\hat{T}(j)\hat{T}(-j-1) = 1, \quad \hat{T}(j)^2 + \hat{T}(-j-1)^2 = 0, \tag{3.37}$$

which has no solution at $j = -1/2$. This shows that there is no analytic wave function for identity B-brane even for integer k.

This result seems in contradiction with the known classification of B-branes in minimal model, where we do have identity B-brane. Naively, Liouville theory would have to have the same set of B-branes as in minimal model when k is sent to a negative integer. This apparent contradiction is due to the consistency with reflection relation we imposed on B-branes. Minimal models are theories without continuous spectrum of representations, and we only consider bulk operators $V^{j(s,\bar{s})}_{m,\bar{m}}$ with $2j \in \mathbb{Z}_{\geq 0}$ and do not care about reflection relations. Classification of B-branes in such models therefore needs a different treatment, and the modular bootstrap analysis should work.

We will not go into any more detail on these special models since irrational models with continuous spectrum are of our main interest.

4. One-point functions on a disc

Here we derive the wave functions for boundary states using Ward identity of disc correlators containing degenerate fields. We will see that all the wave functions for A-branes obtained in previous section satisfy the constraint arising from Ward identity. For B-branes, this is the only way available for obtaining wave functions.

The main idea of this analysis is the application of the techniques invented in [4] to disc correlators. The analysis along this path has been done in Liouville theory in [8] and $N = 1$ super-Liouville theory in [12, 13]. For $N = 2$ Liouville theory, relevant disc correlators have been partially analyzed in [16, 17].

4.1 A-branes

The wave functions U for various A-branes were defined so as to agree with disc one-point structure constants. Namely, the one point function of bulk operators on the upper half plane is given by

$$\langle V^{j(s,\bar{s})}_{m,\bar{m}}(z, \bar{z})\rangle_A = |z - \bar{z}|^{-2h} U_A(j, m, s)\delta_{m,\bar{m}}\delta_{s,\bar{s}}. \tag{4.1}$$

A powerful constraint on U can be derived from the conformal bootstrap of disc two-point function involving degenerate operators. In the following we study those containing $j = 1/2$ or $j = k/2$ degenerate fields. We will use the OPE formulae of bulk operators involving $j = 1/2$ and $j = k/2$ operators (2.66), (2.67), as well as the expressions for reflection coefficients for bulk operators (2.69).

4.1.1 $\langle \mathbf{V}^{1/2}\mathbf{V}^{j} \rangle$ for A-branes

We start with the following correlator

$$\langle V^{1/2}_{n,n}(z_0) V^{j(s,s)}_{m,m}(z_1)\rangle = |z_{0\bar{1}}|^{-4h_0}|z_{1\bar{1}}|^{2h_0-2h_1} F(z), \quad (n = \pm\tfrac{1}{2}, \quad z \equiv \left|\tfrac{z_{01}}{z_{0\bar{1}}}\right|^2) \tag{4.2}$$

where $h_0 = \frac{4n^2-3}{4k}$, $h_1 = \frac{(m+s)^2-j(j+1)}{k} + \frac{s^2}{2}$. $V^{1/2}_{n,n}$ does not satisfy the quantization law for θ-momentum and winding number but is perturbatively well-defined. $F(z)$ is a solution of a certain differential equation that arises from superconformal Ward identity, and is expressed as the following integral

$$F(z) = z^{\frac{2mn-j}{k}}(1-z)^{-\frac{1}{k}} \int dt|t|^{\frac{2j}{k}}|t-z|^{\frac{1}{k}}|t-1|^{\frac{1}{k}} \left\{ \frac{m}{t} + \frac{n}{t-z} - \frac{n}{t-1} \right\}. \tag{4.3}$$

This expression is easily obtained from the free field realization as a correlator with one screening operator (denoted as $\tilde{\mu}\tilde{S}$ previously) inserted. The cross-ratio z takes values in $0 \le z \le 1$, so we divide the real line into four segments

$$(0)\ [-\infty, 0] \quad (1)\ [0, z] \quad (2)\ [z, 1] \quad (3)\ [1, \infty], \tag{4.4}$$

and define $F_i(z)$ by the t-integration over the i-th segment. Then the s-channel basis diagonalizing the monodromy around $z = 0$ is given by F_1 and F_3:

$$\begin{aligned} \frac{k\Gamma(1-\frac{2j+1}{k})}{\{m+2n(j+1)\}\Gamma(\frac{1}{k})\Gamma(-\frac{2j+2}{k})} F_3(z) &= F^s_+(z) \ \sim\ z^{\frac{2mn-j}{k}}, \\ \frac{k\Gamma(1+\frac{2j+1}{k})}{(m-2nj)\Gamma(\frac{2j}{k})\Gamma(\frac{1}{k})} F_1(z) &= F^s_-(z) \ \sim\ z^{\frac{2mn+j+1}{k}}. \end{aligned} \tag{4.5}$$

$F(z)$ in (4.2) should therefore be written in terms of them as

$$F(z) = \sum_{\pm} C_\pm(n, n; j, m, m) U(j \pm \tfrac{1}{2}, m+n, s) F^s_\pm(z). \tag{4.6}$$

The t-channel basis diagonalizing the monodromy around $z = 1$ is given by

$$\begin{aligned} -\frac{\Gamma(-\frac{2}{k})}{m\Gamma(\frac{2j}{k})\Gamma(-\frac{2j+2}{k})} F_0(z) &= F^t_+(z) \ \sim\ (1-z)^{-\frac{1}{k}}, \\ \frac{\Gamma(\frac{2}{k})}{n\Gamma(\frac{1}{k})\Gamma(\frac{1}{k})} F_2(z) &= F^t_-(z) \ \sim\ (1-z)^{\frac{1}{k}}, \end{aligned} \tag{4.7}$$

and the two bases are related via

$$\begin{aligned} F^s_+ &= x_{++}F^t_+ + x_{+-}F^t_-, \\ F^s_- &= x_{-+}F^t_+ + x_{--}F^t_-, \end{aligned} \tag{4.8}$$

$$\begin{aligned} x_{++} &= \frac{2m\Gamma(1-\frac{2j+1}{k})\Gamma(\frac{2}{k})}{\{m+2n(j+1)\}\Gamma(1-\frac{2j}{k})\Gamma(\frac{1}{k})}, & x_{+-} &= -\frac{2nk\Gamma(1-\frac{2j+1}{k})\Gamma(-\frac{2}{k})}{\{m+2n(j+1)\}\Gamma(-\frac{2j+2}{k})\Gamma(-\frac{1}{k})}, \\ x_{-+} &= \frac{2m\Gamma(1+\frac{2j+1}{k})\Gamma(\frac{2}{k})}{(m-2nj)\Gamma(1+\frac{2j+2}{k})\Gamma(\frac{1}{k})}, & x_{--} &= -\frac{2nk\Gamma(1+\frac{2j+1}{k})\Gamma(-\frac{2}{k})}{(m-2nj)\Gamma(\frac{2j}{k})\Gamma(-\frac{1}{k})}. \end{aligned} \tag{4.9}$$

The term proportional to F^t_- in the t-channel represents the operator $V^{1/2}_{n,n}$ approaching the boundary and turning into identity operator:

$$V^{1/2}_{n,n}(z) \to u(n)|z-\bar{z}|^{\frac{1}{k}} + \cdots. \tag{4.10}$$

We thus obtain the following recursion relation for U:

$$u(n)U(j,m,s) = \sum_{\pm} x_{\pm -} C_{\pm}(nn, jmm) U(j \pm \tfrac{1}{2}, m+n, s). \tag{4.11}$$

or more explicitly

$$\begin{aligned}
&\frac{u(n)\Gamma(-\frac{1}{k})}{\nu^{\frac{1}{2}}\Gamma(-\frac{2}{k})} \frac{U(j,m,s)}{U_{[1]}(j,m,s)} \sin\frac{(2j+1)\pi}{k} \\
&= \frac{U(j+\frac{1}{2}, m+n, s)}{U_{[1]}(j+\frac{1}{2}, m+n, s)} \sin\frac{(2j+2)\pi}{k} + \frac{U(j-\frac{1}{2}, m+n, s)}{U_{[1]}(j-\frac{1}{2}, m+n, s)} \sin\frac{2j\pi}{k}.
\end{aligned} \tag{4.12}$$

The wave functions obtained in the previous subsection all satisfy this constraint with

$$u_{[J,M]}(n) = \frac{2\nu^{\frac{1}{2}}\Gamma(-\frac{2}{k})}{\Gamma(-\frac{1}{k})} e^{\frac{4\pi i M n}{k}} \cos\frac{(2J+1)\pi}{k} \tag{4.13}$$

for all branes labelled by $[J,M]$ (non-chiral non-degenerate branes, degenerate branes $[J_{r,s}, M]$, chiral branes $[J,M]^{\pm}$ and degenerate chiral branes $[J,M]^{\mathrm{dc}}$).

4.1.2 $\langle \mathbf{V^{k/2}V^{j}} \rangle$ for A-branes

We can derive another recursion relation from the two-point function involving $j = \frac{k}{2}$ degenerate representation. Consider the following correlator on a disc:

$$\langle V^{k/2}_{n\bar{n}}(z_0) V^{j(s,s)}_{m\bar{m}}(z_1)\rangle = z_{0\bar{1}}^{-2h_0} z_{1\bar{0}}^{-h_0-\bar{h}_0-h_1+\bar{h}_1} z_{1\bar{1}}^{h_0+\bar{h}_0-h_1-\bar{h}_1} z_{0\bar{1}}^{h_0-\bar{h}_0+h_1-\bar{h}_1} F(z). \tag{4.14}$$

where $z = \left|\frac{z_{01}}{z_{0\bar{1}}}\right|^2$ and

$$h_0 = \frac{n^2}{k} - \frac{k+2}{4}, \quad \bar{h}_0 = \frac{\bar{n}^2}{k} - \frac{k+2}{4}, \quad h_1 = \frac{(m+s)^2 - j(j+1)}{k} + \frac{s^2}{2}, \quad \bar{h}_1 = \frac{(\bar{m}+s)^2 - j(j+1)}{k} + \frac{s^2}{2}. \tag{4.15}$$

The conservation of R-charge requires

$$n + m = \bar{n} + \bar{m}. \tag{4.16}$$

The function F is expressed as a contour integral of the form

$$\begin{aligned}
F(z) = z^{\frac{2nm}{k} - j}(1-z)^{-\frac{2n\bar{n}}{k} - \frac{k}{2}} \int_C dw d\hat{w} |w|^{j+m} |w-z|^{\frac{k}{2}+n} |w-1|^{\frac{k}{2}-\bar{n}} \\
\times |\hat{w}|^{j-m} |\hat{w}-z|^{\frac{k}{2}-n} |\hat{w}-1|^{\frac{k}{2}+\bar{n}} |w-\hat{w}|^{-k-1}.
\end{aligned} \tag{4.17}$$

This can easily be derived using free fields and screening operators, and is shown to satisfy the Ward identity. Note that the solution is unique except for the choice of contours: at first sight it would appear that by flipping j to $-j-1$ we would obtain a new solution, but it is actually not the case. As in the previous paragraph, we assume z to take values in $0 \leq z \leq 1$ and divide the real line into four segments. Different contours give different functions, and we denote various functions as follows:

$$F_{1\hat{1}}(z) \leftrightarrow \{0 < w < \hat{w} < z\}, \quad F_{\hat{1}2}(z) \leftrightarrow \{0 < \hat{w} < z < w < 1\}, \quad \text{etc.} \tag{4.18}$$

The basis of contour integrals in the s-channel that diagonalizes the monodromy around $z = 0$ is given by the following six:

$$F_{1\hat{1}},\ F_{\hat{1}1},\ F_{3\hat{3}},\ F_{\hat{3}3},\ F_{1\hat{3}},\ F_{\hat{1}3}, \tag{4.19}$$

but only four linear combinations out of them are indeed the solutions of differential equation. The reason for this is that, since the integrals are along segments, one must always worry about the boundary term when checking that these integrals indeed satisfy a differential equation. A simple way to analyze this is to see whether one can replace the contours ending on points $0, 1, z, \infty$ by those encircling them. For example, $F_{1\hat{3}}$ might fail to satisfy a differential equation due to the boundary $w = 0, w = z$ and $\hat{w} = 1, \hat{w} = \infty$, but this is not the case since one can replace the contours by those not ending on those points. Such replacements of contours are not possible for $F_{1\hat{1}}$ or $F_{\hat{1}1}$, but a certain linear combination of them does have a closed contour integral expression. In this way one finds that there are only four solutions as listed below: (in the following we denote $\mathbf{s}(x) \equiv \sin(\pi x), \mathbf{c}(x) \equiv \cos(\pi x)$)

$$\begin{aligned}
F^s_+ &= -\frac{\Gamma(-j-\frac{k}{2}+m+n)\Gamma(-j-\frac{k}{2}-m-n)\Gamma(-2j)}{\pi\Gamma(-j+\bar{m})\Gamma(-j-\bar{m})\Gamma(-2j-k-1)}\left\{\mathbf{s}(\tfrac{k}{2}-\bar{n})F_{3\hat{3}}+\mathbf{s}(\tfrac{k}{2}+\bar{n})F_{\hat{3}3}\right\},\\
F^s_- &= -\frac{\Gamma(1+j-\frac{k}{2}+m+n)\Gamma(1+j-\frac{k}{2}-m-n)\Gamma(2j+2)}{\pi\Gamma(1+j+m)\Gamma(1+j-m)\Gamma(2j-k+1)}\left\{\mathbf{s}(\tfrac{k}{2}-n)F_{1\hat{1}}+\mathbf{s}(\tfrac{k}{2}+n)F_{\hat{1}1}\right\},\\
F^s_\uparrow &= \frac{\Gamma(2+j+\frac{k}{2}-m-n)\Gamma(1+\frac{k}{2}-j-m-n)}{\Gamma(1+j-m)\Gamma(-j-\bar{m})\Gamma(1+\frac{k}{2}-n)\Gamma(1+\frac{k}{2}-\bar{n})}F_{\hat{1}3},\\
F^s_\downarrow &= \frac{\Gamma(2+j+\frac{k}{2}+m+n)\Gamma(1+\frac{k}{2}-j+m+n)}{\Gamma(1+j+m)\Gamma(-j+\bar{m})\Gamma(1+\frac{k}{2}+n)\Gamma(1+\frac{k}{2}+\bar{n})}F_{1\hat{3}}.
\end{aligned} \tag{4.20}$$

They form the s-channel basis of solutions with the asymptotics

$$F^s_+ \sim z^{\frac{2nm}{k}-j},\quad F^s_- \sim z^{\frac{2nm}{k}+j+1},\quad F^s_\uparrow \sim z^{\frac{2nm}{k}+\frac{k}{2}-m-n+1},\quad F^s_\downarrow \sim z^{\frac{2nm}{k}+\frac{k}{2}+m+n+1}. \tag{4.21}$$

These asymptotics are easily derived by using the function G_k and its properties summarized in the appendix. $F(z)$ in (4.14) should therefore be expressed as

$$\begin{aligned}
e^{-i\pi(h_0+h_1)-\frac{i\pi}{k}(n\bar{m}-\bar{n}m)}F(z) &= \sum_\pm \tilde{C}_\pm(n\bar{n};jm\bar{m})U(j\pm\tfrac{k}{2},m+n,s)F^s_\pm(z)\\
&\quad+\sum_\updownarrow \tilde{C}_\updownarrow(n\bar{n};jm\bar{m})U(j,m+n\mp\tfrac{k}{2}\mp 1,s\pm 1)F^s_\updownarrow(z)\,.
\end{aligned} \tag{4.22}$$

On the other hand, the basis in the t-channel diagonalizing the monodromy around $z = 1$ is given by

$$F_{0\hat{0}},\ F_{\hat{0}0},\ F_{2\hat{2}},\ F_{\hat{2}2},\ F_{0\hat{2}},\ F_{\hat{0}2}. \tag{4.23}$$

The two bases are related as follows:

$$F_{1\hat{1}} = F_{0\hat{0}}\frac{\mathbf{s}(j-m)\mathbf{s}(j+\bar{m})}{\mathbf{s}(k)\mathbf{s}(k+m-\bar{m})} - F_{\hat{0}0}\frac{\mathbf{s}(j-\bar{m})\mathbf{s}(k+j+m)}{\mathbf{s}(k)\mathbf{s}(k+m-\bar{m})}$$

$$+\left\{\mathbf{s}(\tfrac{k}{2}+\bar{n})F_{2\hat{2}}+\mathbf{s}(\tfrac{k}{2}-\bar{n})F_{\hat{2}2}\right\}\frac{\mathbf{s}(\frac{k}{2}+n)}{\mathbf{s}(k)\mathbf{s}(m-\bar{m})}-F_{0\hat{2}}\frac{\mathbf{s}(\frac{k}{2}+\bar{n})\mathbf{s}(j+\bar{m})}{\mathbf{s}(m-\bar{m})\mathbf{s}(k+m-\bar{m})},$$

$$F_{\hat{1}1}=-F_{0\hat{0}}\frac{\mathbf{s}(j+\bar{m})\mathbf{s}(k+j-m)}{\mathbf{s}(k)\mathbf{s}(k-m+\bar{m})}+F_{\hat{0}0}\frac{\mathbf{s}(j-\bar{m})\mathbf{s}(j+m)}{\mathbf{s}(k)\mathbf{s}(k-m+\bar{m})}$$

$$-\left\{\mathbf{s}(\tfrac{k}{2}-\bar{n})F_{\hat{2}2}+\mathbf{s}(\tfrac{k}{2}+\bar{n})F_{2\hat{2}}\right\}\frac{\mathbf{s}(\frac{k}{2}-n)}{\mathbf{s}(k)\mathbf{s}(m-\bar{m})}+F_{\hat{0}2}\frac{\mathbf{s}(\frac{k}{2}-\bar{n})\mathbf{s}(j-\bar{m})}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(m-\bar{m})},$$

$$F_{3\hat{3}}=F_{0\hat{0}}\frac{\mathbf{s}(j-m)\mathbf{s}(j+\bar{m})}{\mathbf{s}(k)\mathbf{s}(k-m+\bar{m})}+F_{\hat{0}0}\frac{\mathbf{s}(j+m)\mathbf{s}(k-j+\bar{m})}{\mathbf{s}(k)\mathbf{s}(k-m+\bar{m})}$$

$$-\left\{\mathbf{s}(\tfrac{k}{2}+n)F_{2\hat{2}}+\mathbf{s}(\tfrac{k}{2}-n)F_{\hat{2}2}\right\}\frac{\mathbf{s}(\frac{k}{2}+\bar{n})}{\mathbf{s}(k)\mathbf{s}(m-\bar{m})}+F_{\hat{0}2}\frac{\mathbf{s}(\frac{k}{2}+n)\mathbf{s}(j-m)}{\mathbf{s}(m-\bar{m})\mathbf{s}(k-m+\bar{m})},$$

$$F_{\hat{3}3}=F_{0\hat{0}}\frac{\mathbf{s}(j-m)\mathbf{s}(k-j-\bar{m})}{\mathbf{s}(k)\mathbf{s}(k+m-\bar{m})}+F_{\hat{0}0}\frac{\mathbf{s}(j+m)\mathbf{s}(j-\bar{m})}{\mathbf{s}(k)\mathbf{s}(k+m-\bar{m})}$$

$$+\left\{\mathbf{s}(\tfrac{k}{2}+n)F_{2\hat{2}}+\mathbf{s}(\tfrac{k}{2}-n)F_{\hat{2}2}\right\}\frac{\mathbf{s}(\frac{k}{2}-\bar{n})}{\mathbf{s}(k)\mathbf{s}(m-\bar{m})}-F_{0\hat{2}}\frac{\mathbf{s}(\frac{k}{2}-n)\mathbf{s}(j+m)}{\mathbf{s}(m-\bar{m})\mathbf{s}(k+m-\bar{m})},$$

$$F_{1\hat{3}}=F_{\hat{0}0}\frac{\mathbf{s}(j+m)\mathbf{s}(j-\bar{m})-\mathbf{s}(k-j+m)\mathbf{s}(k+j+\bar{m})}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(k+m-\bar{m})}-F_{0\hat{0}}\frac{2\mathbf{c}(k)\mathbf{s}(j-m)\mathbf{s}(j+\bar{m})}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(k+m-\bar{m})}$$

$$-F_{0\hat{2}}\frac{\mathbf{s}(j+\bar{m})\mathbf{s}(\frac{k}{2}-n)}{\mathbf{s}(m-\bar{m})\mathbf{s}(k+m-\bar{m})}+F_{\hat{0}2}\frac{\mathbf{s}(j-m)\mathbf{s}(\frac{k}{2}-\bar{n})}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(m-\bar{m})},$$

$$F_{\hat{1}3}=F_{0\hat{0}}\frac{\mathbf{s}(j+\bar{m})\mathbf{s}(j-m)-\mathbf{s}(k-j-m)\mathbf{s}(k+j-\bar{m})}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(k+m-\bar{m})}-F_{\hat{0}0}\frac{2\mathbf{c}(k)\mathbf{s}(j-\bar{m})\mathbf{s}(j+m)}{\mathbf{s}(k-m+\bar{m})\mathbf{s}(k+m-\bar{m})}$$

$$+F_{\hat{0}2}\frac{\mathbf{s}(\frac{k}{2}+n)\mathbf{s}(j-\bar{m})}{\mathbf{s}(m-\bar{m})\mathbf{s}(k-m+\bar{m})}-F_{0\hat{2}}\frac{\mathbf{s}(\frac{k}{2}+\bar{n})\mathbf{s}(j+m)}{\mathbf{s}(m-\bar{m})\mathbf{s}(k+m-\bar{m})}. \tag{4.24}$$

The t-channel describes the degenerate operator $V_{n\bar{n}}^{k/2}$ approaching the boundary and decomposing into a sum of boundary operators. To derive a recursion relation for the one-point structure constants, we would first be interested in the terms proportional to the boundary $j=0$ operator. They should behave like $\sim (1-z)^{-\frac{2n\bar{n}}{k}+\frac{k}{2}+1}$ and are proportional to $F_{2\hat{2}}$ or $F_{\hat{2}2}$. Very surprisingly, those functions do not appear when we express the functions $F^s_{\pm}, F^s_{\updownarrow}$ in terms of t-channel basis. Therefore we have to focus on the terms proportional to

$$F^t_{\uparrow}=\frac{\Gamma(1+m-\bar{m})\Gamma(k+m-\bar{m}+2)F_{0\hat{2}}}{\Gamma(j+m+1)\Gamma(-j-\bar{m})\Gamma(\frac{k}{2}-n+1)\Gamma(\frac{k}{2}+\bar{n}+1)}\ \sim\ (1-z)^{-\frac{2n\bar{n}}{k}+\frac{k}{2}-n+\bar{n}+1},$$

$$F^t_{\downarrow}=\frac{\Gamma(1-m+\bar{m})\Gamma(k-m+\bar{m}+2)F_{\hat{0}2}}{\Gamma(j-m+1)\Gamma(-j+\bar{m})\Gamma(\frac{k}{2}+n+1)\Gamma(\frac{k}{2}-\bar{n}+1)}\ \sim\ (1-z)^{-\frac{2n\bar{n}}{k}+\frac{k}{2}+n-\bar{n}+1}, \tag{4.25}$$

which correspond to the terms in the self-OPE

$$V_{n\bar{n}}^{k/2}(z)\to|z-\bar{z}|^{-\frac{2n\bar{n}}{k}+\frac{k}{2}-n+\bar{n}+1}\tilde{u}^{\uparrow}(n,\bar{n})B^{\frac{k}{2}(1)}_{n-\bar{n}-\frac{k}{2}-1}(z)$$
$$+|z-\bar{z}|^{-\frac{2n\bar{n}}{k}+\frac{k}{2}+n-\bar{n}+1}\tilde{u}^{\downarrow}(n,\bar{n})B^{\frac{k}{2}(-1)}_{n-\bar{n}+\frac{k}{2}+1}(z)+\cdots \tag{4.26}$$

We will also have to consider the bulk-to-boundary propagators for some special case:

$$\langle B_n^{k/2(\pm1)}(x)V_{m\bar{m}}^{j(s,s)}(z)\rangle=|z-\bar{z}|^{h_2-h_1-\bar{h}_1}(x-z)^{\bar{h}_1-h_1-h_2}(x-\bar{z})^{h_1-\bar{h}_1-h_2}$$
$$\times\delta(m-\bar{m}+n\pm\tfrac{k}{2}\pm1)U^{\pm}(j,m,\bar{m}). \tag{4.27}$$

where $h_1, \bar{h}_1$ are the same as before and $h_2 = \frac{(n\pm 1)^2}{k} - \frac{k}{4}$. The s/t-channel bases are related as follows:

$$\begin{aligned}
F_+^s &= x_{+\uparrow}\Gamma(-m+\bar{m})F_\uparrow^t + x_{+\downarrow}\Gamma(m-\bar{m})F_\downarrow^t + \cdots, \\
F_-^s &= x_{-\uparrow}\Gamma(-m+\bar{m})F_\uparrow^t + x_{-\downarrow}\Gamma(m-\bar{m})F_\downarrow^t + \cdots, \\
F_\uparrow^s &= x_{\uparrow\uparrow}\Gamma(-m+\bar{m})F_\uparrow^t + x_{\uparrow\downarrow}\Gamma(m-\bar{m})F_\downarrow^t + \cdots, \\
F_\downarrow^s &= x_{\downarrow\uparrow}\Gamma(-m+\bar{m})F_\uparrow^t + x_{\downarrow\downarrow}\Gamma(m-\bar{m})F_\downarrow^t + \cdots,
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
x_{+\uparrow} &= \frac{\Gamma(-1-k-m+\bar{m})\Gamma(-j-\frac{k}{2}+m+n)\Gamma(-j-\frac{k}{2}-m-n)\Gamma(-2j)}{\Gamma(-j-m)\Gamma(-j+\bar{m})\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-\bar{n})\Gamma(-2j-k-1)}, \\
x_{-\uparrow} &= \frac{\Gamma(-1-k-m+\bar{m})\Gamma(1+j-\frac{k}{2}+m+n)\Gamma(1+j-\frac{k}{2}-m-n)\Gamma(2j+2)}{\Gamma(1+j-m)\Gamma(1+j+\bar{m})\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-\bar{n})\Gamma(2j-k+1)}, \\
x_{\uparrow\uparrow} &= \frac{\Gamma(-1-k-m+\bar{m})\Gamma(2+j+\frac{k}{2}-m-n)\Gamma(1+\frac{k}{2}-j-m-n)}{\Gamma(1+j-m)\Gamma(-j-m)\Gamma(1+\frac{k}{2}-\bar{n})\Gamma(-\frac{k}{2}-\bar{n})}, \\
x_{\downarrow\uparrow} &= \frac{\Gamma(-1-k-m+\bar{m})\Gamma(2+j+\frac{k}{2}+m+n)\Gamma(1+\frac{k}{2}-j+m+n)}{\Gamma(1+j+\bar{m})\Gamma(-j+\bar{m})\Gamma(1+\frac{k}{2}+n)\Gamma(-\frac{k}{2}+n)}.
\end{aligned} \tag{4.29}$$

The coefficients $x_{\pm\downarrow}, x_{\updownarrow\downarrow}$ are obtained from $x_{\pm\uparrow}, x_{\updownarrow\uparrow}$ by the exchange $m \leftrightarrow \bar{m}$, $n \leftrightarrow \bar{n}$.

The basis change law becomes singular when $m-\bar{m}$ is an integer, which is actually the case for all the perturbatively well-defined vertex operators $V_{m\bar{m}}^{j(s,s)}$. As a consequence, the solutions of differential equation develop a logarithm at $z \sim 1$ and signal the emergence of a logarithmic operator on the boundary. This logarithm can be understood in the following way. Let us focus on the case $m = \bar{m}$, and recall that the boundary operators $B_m^{k/2(\pm1)}$ with $m = \mp(\frac{k}{2}+1)$ are expected from the representation theory to behave like identity. Looking at the basis change law above, it is expected that $B_{m\mp\frac{k}{2}\mp1}^{k/2(\pm1)}$ approaches identity as $m \to 0$ with divergent coefficient:

$$B_{m\mp\frac{k}{2}\mp1}^{k/2(\pm1)} \overset{m\to 0}{\to} c^{\updownarrow}\Gamma(\pm m) \times \mathbf{1}. \tag{4.30}$$

Since we have two sets of operators (both parametrized by m) approaching the identity as $m \to 0$, one can define the logarithmic operator by their difference. This is analogous to the case of the free boson theory of ϕ, where we have continuously many primary operators $e^{ia\phi}$. $e^{\pm ia\phi}$ have the same conformal weight $h = \frac{a^2}{2}$, except at $h = 0$ we have two operators 1 and ϕ, the latter of which is logarithmic and is obtained by a-derivative.

The above argument also shows that the bulk-boundary propagators (4.27) become proportional to $U(j,m)$ when $m = \bar{m}$:

$$U^{\pm}(j,m,\bar{m}) \overset{m\to\bar{m}}{\to} c^{\updownarrow}\Gamma(\mp m \pm \bar{m})U(j,m). \tag{4.31}$$

Using this together with (4.23), (4.29) we obtain another relation between one-point structure constants:

$$\begin{aligned}
c^{\updownarrow}\tilde{u}^{\updownarrow}(n,n)U(j,m,s) = \;& x_{+\updownarrow}\tilde{C}_+(nn;jmm)U(j+\tfrac{k}{2}, m+n, s) \\
&+ x_{-\updownarrow}\tilde{C}_-(nn;jmm)U(j-\tfrac{k}{2}, m+n, s) \\
&+ x_{\uparrow\updownarrow}\tilde{C}_\uparrow(nn;jmm)U(j, m+n-\tfrac{k}{2}-1, s+1) \\
&+ x_{\downarrow\updownarrow}\tilde{C}_\downarrow(nn;jmm)U(j, m+n+\tfrac{k}{2}+1, s-1).
\end{aligned} \tag{4.32}$$

or more explicitly,

$$\begin{aligned}
&c^{\updownarrow}\tilde{u}^{\updownarrow}(n,n)\frac{\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-n)}{k\nu^{\frac{k}{2}}\Gamma(-1-k)}\times\hat{U}(j,m,s)\\
&=-\hat{U}(j+\tfrac{k}{2},m+n,s)+\hat{U}(j,m+n-\tfrac{k}{2}-1,s+1)\\
&\quad-\hat{U}(j-\tfrac{k}{2},m+n,s)+\hat{U}(j,m+n+\tfrac{k}{2}+1,s-1),\\
&\hat{U}(j,m,s) \;=\; \frac{U(j,m,s)\mathbf{s}(2j)}{U_{[1]}(j,m,s)\mathbf{s}(j+m)\mathbf{s}(j-m)}. \qquad (4.33)
\end{aligned}$$

The wave functions obtained in the previous section all satisfy this equation. The structure constants $c^{\updownarrow}, \tilde{u}^{\updownarrow}(n,n)$ satisfy

$$c^{\updownarrow}_{[J,M]}\tilde{u}^{\updownarrow}_{[J,M]}(n,n)=4\sin\{(J+M)\pi\}\sin\{(J-M)\pi\}\frac{e^{\frac{4\pi iMn}{k}}k\nu^{\frac{k}{2}}\Gamma(-1-k)}{\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-n)} \qquad (4.34)$$

for all A-branes. Note that it vanishes for the A-branes corresponding to chiral representations $[J,M]^{\pm}$ and $[J,M]^{\rm dc}$.

4.2 B-branes

The disc one-point function for B-branes takes the form

$$\langle V^{j(s,\bar{s})}_{m,\bar{m}}(z)\rangle_B=|z-\bar{z}|^{-2h}T(j,m,s)\delta_{m+\bar{m},0}\delta_{s+\bar{s},0} \qquad (4.35)$$

where $m+\bar{m}=s+\bar{s}=0$ follows from $L_0-\bar{L}_0=J_0+\bar{J}_0=0$. The functional form of T is largely determined from symmetry argument. First, since s is the momentum of bosonization of conserved $U(1)$ current, it is reasonable to assume its s-dependence to be simply e^{ias} for some constant a. Then the boundary condition on supercurrents $T^{\pm}_F=\bar{T}^{\pm}_F$ require

$$T(j,m,s)=e^{ia(m+s)}\nu^{j+\frac{1}{2}}\frac{\Gamma(-2j)\Gamma(-\frac{2j+1}{k})}{\Gamma(-j+m)\Gamma(-j-m)}T_0(j,m) \qquad (4.36)$$

where some functions of j were put for later convenience, and T_0 is periodic in m with unit period. Noticing that $m\in\frac{1}{2}\mathbb{Z}$ for perturbatively well-defined operators, one finds such periodic functions are proportional either to 1 or $e^{2\pi im}$. Writing $T_0(j,m)=\hat{T}(j)+e^{2\pi im}\check{T}(j)$, one finds $\check{T}(j)=\hat{T}(-j-1)$ from the reflection relation. Thus all we are left with is to determine an unknown function $\hat{T}(j)$ in the *ansatz*

$$T(j,m,s)=e^{ia(m+s)}\nu^{j+\frac{1}{2}}\frac{\Gamma(-2j)\Gamma(-\frac{2j+1}{k})}{\Gamma(-j+m)\Gamma(-j-m)}\{\hat{T}(j)+e^{2\pi im}\hat{T}(-j-1)\}. \qquad (4.37)$$

One might think of other ansaetze, but they all reduce to the above one under the condition $m\in\frac{1}{2}\mathbb{Z}$. For example, the ansatz

$$T(j,m,s)=e^{ia(m+s)}\nu^{j+\frac{1}{2}}\frac{\Gamma(1+j+m)\Gamma(1+j-m)}{\Gamma(2j+2)\Gamma(\frac{2j+1}{k})}\{\hat{f}(j)+e^{2\pi im}\hat{f}(-j-1)\} \qquad (4.38)$$

is related to the previous ansatz by

$$\hat{T}(j) \pm \hat{T}(-j-1) = -\frac{2}{k}\frac{\mathbf{s}(2j)\mathbf{s}(\frac{2j+1}{k})}{\mathbf{c}(2j) \mp 1}\{f(j) \pm f(-j-1)\}. \tag{4.39}$$

The analysis of the disc correlators for B-branes proceeds in a similar way as for A-branes. First, the disc two-point function containing $j = 1/2$ degenerate operator takes the form

$$\langle V^{1/2}_{n,-n}(z_0) V^{j(s,-s)}_{m,-m}(z_1)\rangle = |z_{0\bar{1}}|^{-4h_0}|z_{1\bar{1}}|^{2h_0-2h_1}F(z), \quad (n = \pm\tfrac{1}{2}, \quad z \equiv \left|\tfrac{z_{01}}{z_{0\bar{1}}}\right|^2) \tag{4.40}$$

where $F(z)$ is the same contour integral expression as for A-branes,

$$F(z) = z^{\frac{2mn-j}{k}}(1-z)^{-\frac{1}{k}}\int dt|t|^{\frac{2j}{k}}|t-z|^{\frac{1}{k}}|t-1|^{\frac{1}{k}}\left\{\frac{m}{t} + \frac{n}{t-z} - \frac{n}{t-1}\right\}. \tag{4.41}$$

Using the same bases $F^s_\pm$ as before, one finds

$$\begin{aligned} F(z) &= \sum_\pm C_\pm(n,-n;j,m,-m)T(j \pm \tfrac{1}{2}, m+n, s)F^s_\pm(z) \\ &= t(n)T(j,m,s)F^t_-(z) + \cdots, \end{aligned} \tag{4.42}$$

where $t(n)$ is the self OPE coefficient of $V^{1/2}_{n,-n}$ turning into boundary identity operator,

$$V^{1/2}_{n,-n}(z) \to t(n)|z-\bar{z}|^{\frac{1}{k}} + \cdots. \tag{4.43}$$

We thus obtain a recursion relation for T:

$$t(n)T(j,m,s) = \sum_\pm x_{\pm -}C_\pm(n,-n;j,m,-m)T(j \pm \tfrac{1}{2}, m+n, s), \tag{4.44}$$

where $x_{\pm-}$ are given in (4.9). In terms of $\hat{T}$ it becomes simple. Introducing

$$\hat{T}_\pm = \hat{T}(j) \pm \hat{T}(-j-1) \tag{4.45}$$

one finds

$$q\hat{T}_\pm(j) = \hat{T}_\mp(j+\tfrac{1}{2}) - \hat{T}_\mp(j-\tfrac{1}{2}), \quad q = \frac{t(n)\Gamma(-\frac{1}{k})}{e^{ian}\nu^{\frac{1}{2}}\Gamma(-\frac{2}{k})}. \tag{4.46}$$

Next, the two-point function containing $j = k/2$ operator is

$$\langle V^{k/2}_{n,-\bar{n}}(z_0) V^{j(s,-s)}_{m,-\bar{m}}(z_1)\rangle = z_{0\bar{1}}^{-2h_0} z_{1\bar{0}}^{-h_0-\bar{h}_0-h_1+\bar{h}_1} z_{1\bar{1}}^{h_0+\bar{h}_0-h_1-\bar{h}_1} z_{\bar{0}\bar{1}}^{h_0-\bar{h}_0+h_1-\bar{h}_1}F(z), \tag{4.47}$$

where $z = \left|\frac{z_{01}}{z_{0\bar{1}}}\right|^2$ and $F(z)$ is the same contour integral (4.17) as was given for A-branes. $F(z)$ should be expressed in terms of s-channel basis as

$$e^{-i\pi(h_0+h_1)+\frac{i\pi}{k}(n\bar{m}-\bar{n}m)}F(z) = \sum_\pm \tilde{C}_\pm(n,-\bar{n};j,m,-\bar{m})T(j \pm \tfrac{k}{2}, m+n, s)F^s_\pm(z), \tag{4.48}$$

because the one-point function vanishes for operators with $s + \bar{s} \neq 0$ and therefore we cannot have terms proportional to $\tilde{C}_\updownarrow$ in the right hand side. As before, after rewriting

$F(z)$ in t-channel basis we focus on the terms proportional to $F^t_{\updownarrow}$ which correspond to the following terms in the self-OPE

$$V^{k/2}_{n,-\bar{n}}(z) \to |z-\bar{z}|^{-\frac{2n\bar{n}}{k}+\frac{k}{2}-n+\bar{n}+1}\tilde{t}^{\uparrow}(n,-\bar{n})B^{k/2(1)}_{n-\bar{n}-\frac{k}{2}-1}(z)$$
$$+|z-\bar{z}|^{-\frac{2n\bar{n}}{k}+\frac{k}{2}+n-\bar{n}+1}\tilde{t}^{\downarrow}(n,-\bar{n})B^{k/2(-1)}_{n-\bar{n}+\frac{k}{2}+1}(z)+\cdots \tag{4.49}$$

The two-point function exhibits a logarithmic behavior at $z=1$ when $n-\bar{n}=0$, and we interpret it as the degeneracy of the following boundary operators:

$$B^{k/2(\pm1)}_{m\mp\frac{k}{2}\mp1} \stackrel{m\to0}{\longrightarrow} c^{\updownarrow}\Gamma(\mp m)\times\mathbf{1}. \tag{4.50}$$

Thus we obtain a recursion relation:

$$c^{\updownarrow}\tilde{t}^{\updownarrow}(n,-n)T(j,m,s) = x_{+\updownarrow}\tilde{C}_+(n,-n;j,m,-m)T(j+\tfrac{k}{2},m+n,s)$$
$$+x_{-\updownarrow}\tilde{C}_-(n,-n;j,m,-m)T(j-\tfrac{k}{2},m+n,s) \tag{4.51}$$

where $x_{\pm\updownarrow}$ are the ones given in (4.29). In terms of $\hat{T}(j)$ this can be rewritten as

$$p_0\hat{T}_\pm(j)=\hat{T}_\pm(j+\tfrac{k}{2})+\hat{T}_\pm(j-\tfrac{k}{2}),\quad p_1\hat{T}_\pm(j)=\hat{T}_\mp(j+\tfrac{k}{2})-\hat{T}_\mp(j-\tfrac{k}{2}), \tag{4.52}$$

where

$$p_{0,1}=\left.\frac{c^{\updownarrow}\tilde{t}^{\updownarrow}(n,-n)\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-n)}{e^{ian}k\nu^{\frac{k}{2}}\Gamma(-k-1)}\right|_{2n=\text{even,odd}} \tag{4.53}$$

A one-parameter family of solutions to (4.46), (4.52) can be found easily:

$$\hat{T}(j)=\exp(2j+1)u,\quad q=2\sinh u,\quad p_0=2\cosh ku,\quad p_1=2\sinh ku. \tag{4.54}$$

Using labels $[J,M]$ instead of (a,u), we summarize the result for B-branes below.

$$T_{[J,M]}(j,m,s) = T_0\nu^{j+\frac{1}{2}}e^{\frac{4\pi iM}{k}(m+s)}\frac{\Gamma(-2j)\Gamma(-\frac{2j+1}{k})}{\Gamma(-j+m)\Gamma(-j+m)}$$
$$\times\left\{e^{\frac{i\pi}{k}(2j+1)(2J+1)}+e^{2\pi im}e^{-\frac{i\pi}{k}(2j+1)(2J+1)}\right\},$$
$$t(n) = 2ie^{\frac{4\pi iMn}{k}}\nu^{\frac{1}{2}}\frac{\Gamma(-\frac{2}{k})}{\Gamma(-\frac{1}{k})}\sin\{\tfrac{\pi}{k}(2J+1)\},$$
$$c^{\updownarrow}\tilde{t}^{\updownarrow}(n,-n) = \frac{e^{\frac{4\pi iMn}{k}}k\nu^{\frac{k}{2}}\Gamma(-k-1)}{\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-n)}(e^{i\pi(2J+1)}+e^{2\pi in}e^{-i\pi(2J+1)}). \tag{4.55}$$

5. Boundary interactions

Here we discuss the Lagrangian description of various boundary states and possible boundary interactions in $N=2$ Liouville theory. Some aspects of this issue have been studied in [15 – 17].

In general $N = (2,2)$ Landau-Ginzburg models defined by the action

$$S = \int d^2z d^4\theta K(\Phi^i, \bar{\Phi}^i) + \int d^2z d\theta^+ d\bar{\theta}^+ W(\Phi^i) + \int d^2z d\theta^- d\bar{\theta}^- \bar{W}(\bar{\Phi}^i), \tag{5.1}$$

there are several ways to preserve supersymmetry on worldsheets with boundary[33 − 38]. One way is to put boundary conditions on fields Φ^j; the boundary states are then naturally associated to the submanifolds of Φ^j-space defined by the boundary conditions. A-branes are Lagrangian submanifolds which should also be pre-images of a straight line in complex W-plane, whereas B-branes are holomorphic submanifolds which are level-sets of W [33, 34]. More recently it has been found that the *matrix factorization* enables one to describe B-branes in LG models in terms of certain boundary interactions which involve a Chan Paton degree of freedom [37, 38]. In this section we first propose the form of boundary interaction for B-branes using this approach, and reproduce a few disc structure constants obtained in the previous section from perturbative computation. Then we make a similar proposal for A-branes.

5.1 B-branes

Consider as a LG theory of a single chiral field Φ, and assume the superpotential W factorizes as $W(\Phi) = \frac{1}{2}f(\Phi)g(\Phi)$. Then on the B-boundary[35] defined by

$$z = \bar{z}, \quad \theta^+ = \bar{\theta}^+, \quad \theta^- = \bar{\theta}^-, \tag{5.2}$$

one introduces the boundary supercovariant derivative

$$D^B_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^\mp\partial_x, \qquad (x \equiv \mathrm{Re}(z)) \tag{5.3}$$

and the fermionic superfields $\Gamma, \bar{\Gamma}$ satisfying

$$D^B_-\Gamma = g(\Phi), \quad D^B_+\bar{\Gamma} = \bar{g}(\bar{\Phi}), \tag{5.4}$$

in terms of which the boundary interaction is expressed in the following way:

$$S_{\text{boundary}} = -\frac{1}{2}\oint dx \left[\int d\theta^+ d\theta^- \bar{\Gamma}\Gamma + \int d\theta^+ \Gamma f(\Phi) + \int d\theta^- \bar{\Gamma}\bar{f}(\bar{\Phi})\right]. \tag{5.5}$$

Using the θ-expansion

$$\begin{aligned} \Gamma &= \lambda + \theta^- g(\phi) + \theta^+ G - i\theta^+\theta^- \left\{\partial_x\lambda + \sqrt{2}g'(\phi)(\psi_+ + \bar{\psi}_+)\right\}, \\ \bar{\Gamma} &= \bar{\lambda} + \theta^+ \bar{g}(\bar{\phi}) + \theta^- \bar{G} + i\theta^+\theta^- \left\{\partial_x\bar{\lambda} + \sqrt{2}\bar{g}'(\bar{\phi})(\psi_- + \bar{\psi}_-)\right\} \end{aligned} \tag{5.6}$$

the boundary interaction can be rewritten as follows

$$\begin{aligned} \mathcal{L}_{\text{boundary}} = &-i\bar{\lambda}\dot{\lambda} + \tfrac{1}{2}|g|^2 - \tfrac{1}{2}|G|^2 - \tfrac{1}{2}Gf - \tfrac{1}{2}\bar{G}\bar{f} \\ &+ \tfrac{i}{\sqrt{2}}\{\lambda f' - \bar{\lambda}g'\}(\psi_+ + \bar{\psi}_+) + \tfrac{i}{\sqrt{2}}\{\bar{\lambda}\bar{f}' - \lambda\bar{g}'\}(\psi_- + \bar{\psi}_-). \end{aligned} \tag{5.7}$$

The B-type supersymmetry variation of S_{boundary} precisely cancels the surface term arising from the variation of the bulk action. It is easy to rotate the boundary condition by R-symmetry, although we have not taken it into account explicitly.

We apply this prescription to the B-type boundary states in $N = 2$ Liouville theory. At first sight, non-trivial factorizations break the invariance under the unit period shift of θ, but the theory remains invariant if we let the boundary fermions $\lambda, \bar{\lambda}$ (or more precisely all the components in the superfields $\Gamma, \bar{\Gamma}$) transform as $g(\Phi), \bar{g}(\bar{\Phi})$. We will only consider the cases with

$$f \sim g \sim W^{1/2},$$

because a copy of $N = 2$ superconformal symmetry is unbroken only for this choice [15]. Note also that the boundary interactions then become precisely the *holomorphic square roots* of bulk interactions, and the theory can still be regarded as a perturbed free CFT after the irrelevant terms are discarded. After suitably normalizing boundary fields and incorporating the effects of nonzero worldsheet curvature, the boundary action can be written as follows:

$$\begin{aligned} &\oint dx \left[\bar{\lambda}\partial_x\lambda - \sqrt{\tfrac{2}{k}}\tfrac{K\rho}{4\pi}\right] + \mu_B S_B + \mu_{\bar{B}} S_{\bar{B}} + \bar{\mu}_B \bar{S}_B + \bar{\mu}_{\bar{B}} \bar{S}_{\bar{B}} \\ &= \oint dx \left[\bar{\lambda}\partial_x\lambda - \sqrt{\tfrac{2}{k}}\tfrac{K\rho}{4\pi} - (\mu_B\lambda + \mu_{\bar{B}}\bar{\lambda})e^{-\sqrt{\frac{k}{2}}\phi_L + iH_L} - e^{-\sqrt{\frac{k}{2}}\bar{\phi}_L - iH_L}(\bar{\mu}_B\lambda + \bar{\mu}_{\bar{B}}\bar{\lambda})\right], \end{aligned} \tag{5.8}$$

where K denotes the curvature of the boundary appearing in the Euler number formula

$$\chi = 2 - 2\sharp(\text{handles}) - \sharp(\text{holes}) = \int_\Sigma \frac{\sqrt{g}R}{4\pi} + \int_{\partial\Sigma} \frac{K}{2\pi}. \tag{5.9}$$

The terms proportional to $\mu_B, \bar{\mu}_B$, etc will be called the boundary screening operators. From the condition $fg = 2W$ one finds

$$\mu_B \mu_{\bar{B}} \;=\; \bar{\mu}_B \bar{\mu}_{\bar{B}} \;=\; \frac{\mu k}{2\pi}. \tag{5.10}$$

Note that the boundary fermions were renormalized to have the standard propagator,

$$\langle \lambda(x)\bar{\lambda}(x')\rangle \;=\; \langle \bar{\lambda}(x)\lambda(x')\rangle \;=\; \frac{1}{2}\text{sign}(x - x'). \tag{5.11}$$

It also follows from this that any non-vanishing correlator of $\lambda, \bar{\lambda}$ is taking values $\pm 1/2$, e.g.,

$$\begin{aligned} &\langle \lambda(x_1)\bar{\lambda}(x_{\bar{1}}) \cdots \lambda(x_n)\bar{\lambda}(x_{\bar{n}})\rangle \\ &= \langle \bar{\lambda}(x_1)\lambda(x_{\bar{1}}) \cdots \bar{\lambda}(x_n)\lambda(x_{\bar{n}})\rangle = \frac{1}{2} \quad (x_1 > x_{\bar{1}} > \cdots > x_n > x_{\bar{n}}). \end{aligned} \tag{5.12}$$

The boundary fermions introduce the Chan-Paton degree of freedom on each boundary. For example, the Hamiltonian quantization of the theory on the strip $(0 \le \sigma \le \pi,\ \tau \in \mathbb{R})$ has two sets of fermions $\lambda_0(\tau), \bar{\lambda}_0(\tau)$ and $\lambda_\pi(\tau), \bar{\lambda}_\pi(\tau)$, which under the free field approximation satisfy the standard anti-commutation relation. So the Chan-Paton space is two-dimensional for each boundary, and is spanned by $|0\rangle$ and $|1\rangle = \bar{\lambda}|0\rangle$ where $|0\rangle$ is annihilated by λ.

JHEP12(2006)061

From the viewpoint of perturbed free CFT, one can also consider the following interaction:

$$\tilde{\mu}_B \tilde{S}_B \;=\; -\,\tilde{\mu}_B \oint dx(\lambda\bar{\lambda} - \bar{\lambda}\lambda)(\psi_+\psi_- - i\sqrt{2k}\partial\theta)e^{-\sqrt{\frac{2}{k}}\rho_L} \tag{5.13}$$

which is also a holomorphic square root of a bulk screening operator. The above operator depends on boundary fermions in a strange manner, but the reason will be explained shortly.

5.1.1 Computation of disc correlators

The relations between the labels of boundary states and the boundary couplings can be obtained by computing some disc structure constants from free field approach. Let us begin by setting up the consistent rules for computing correlators on the upper half plane. Namely, we need to be able to calculate correlators so that they are either invariant or flipping sign under re-orderings of operators appearing in a correlator $\langle\cdots\rangle$, when all the operators are in the physical spectrum and their Grassmann parity is suitably defined.

The Neumann boundary conditions on the fields ρ, θ, H correlate the left- and right-moving sectors of the theory. We evaluate it using the propagators,

$$\langle\rho_L(z)\rho_R(\bar{z}')\rangle \;=\; \langle\theta_L(z)\theta_R(\bar{z}')\rangle \;=\; \langle H_L(z)H_R(\bar{z}')\rangle \;=\; -\ln(z-\bar{z}') + \frac{i\pi}{2}. \tag{5.14}$$

The Wick contraction of free fields gives, after taking the factor (2.41) into account, the following rule for correlators of bulk operators:

$$\langle \textstyle\prod_i V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z_i,\bar{z}_i)\rangle \;\sim\; \prod_i |z_i-\bar{z}_i|^{\gamma_{i\bar{i}}} \prod_{i<j}(z_i-z_j)^{\gamma_{ij}}(z_i-\bar{z}_j)^{\gamma_{i\bar{j}}}(\bar{z}_i-z_j)^{\gamma_{\bar{i}j}}(\bar{z}_i-\bar{z}_j)^{\gamma_{\bar{i}\bar{j}}}, \tag{5.15}$$

where $\gamma_{ij} \equiv \frac{2}{k}\{(m_i+s_i)(m_j+s_j) - j_i j_j\} + s_i s_j$, etc. The Wick contraction involving boundary operators is defined simply by

$$\begin{aligned} V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z)B^{j_a(s_a)}_{m_a}(x) &\sim (z-x)^{\gamma_{ia}}(\bar{z}-x)^{\gamma_{\bar{i}a}}, \\ B^{j_a(s_a)}_{m_a}(x)V^{j_i(s_i,\bar{s}_i)}_{m_i,\bar{m}_i}(z) &\sim (x-z)^{\gamma_{ia}}(x-\bar{z})^{\gamma_{\bar{i}a}}, \\ B^{j_a(s_a)}_{m_a}(x)B^{j_b(s_b)}_{m_b}(x') &\sim (x-x')^{\gamma_{ab}}. \end{aligned} \tag{5.16}$$

The first and the second lines generically differ by phase. To ensure $V(z)B(x) = \pm B(x)\times V(z)$, we therefore require that the operator $V^{j(s,\bar{s})}_{m,\bar{m}}$ appearing in the correlator $\langle\cdots\rangle$ between two boundary operators connected by the boundary state $B^{\alpha}_{[J,M]}$ yields a phase factor[3],

$$\exp\left(\frac{2\pi i(M+\alpha)}{k}(m+s-\bar{m}-\bar{s}) + i\pi\alpha(s-\bar{s})\right). \tag{5.17}$$

In addition, we require that the boundary operator $B^{j(s)}_{m}$ connecting the boundary states $B^{\alpha}_{[J,M]}$ and $B^{\alpha'}_{[J',M']}$ satisfy

$$m+s \in M-M'+\alpha-\alpha'+\frac{k}{2}N, \quad\; s\in\alpha-\alpha'+S. \quad (N,S\in\mathbb{Z}) \tag{5.18}$$

[3]If the correlator contain no boundary operators, one takes the unique boundary state appearing on the boundary and consider the similar phase factor.

We require that the physical operators with even (odd) N are accompanied by even (odd) numbers of boundary fermions, and define the Grassmann parity of the boundary operator to be $N + S$ mod 2. Note that this rule makes all the boundary interaction terms in (5.8) Grassmann-even. On the other hand, the Grassmann parity of the bulk operator $V^{j(s,\bar{s})}_{m,\bar{m}}$ is $s - \bar{s}$ mod 2. In order to get the (anti-)commutativity in accordance with this assignment of Grassmann parity, we have to require that the bulk operator $V^{j(s,\bar{s})}_{m,\bar{m}}$ in disc correlators should behave like

$$e^{\frac{i\pi}{2}(m-\bar{m})}\lambda\bar{\lambda} + e^{-\frac{i\pi}{2}(m-\bar{m})}\bar{\lambda}\lambda. \tag{5.19}$$

Finally, the boundary interaction $\tilde{\mu}_B\tilde{S}_B$ of (5.13) has to be proportional to $\lambda\bar{\lambda} - \bar{\lambda}\lambda$ in order to commute with other boundary interactions.

Let us compute some disc structure constants using the free field prescription. To begin with, we compute the coefficient $t(n)$ appearing in

$$V^{1/2}_{n,-n}(z,\bar{z}) \to t(n)|z-\bar{z}|^{\frac{1}{k}} + \cdots. \tag{5.20}$$

It is given by a free field correlator with one insertion of the boundary screening operator $\tilde{\mu}_B\tilde{S}_B$,

$$t(n) \;=\; \langle B_0^{-1}\cdot V^{1/2}_{n,-n}\cdot(-\tilde{\mu}_B\tilde{S}_B)\rangle_{\rm free} \;=\; -\frac{4k\pi\tilde{\mu}_B\Gamma(-\frac{2}{k})}{\Gamma(-\frac{1}{k})^2}e^{\frac{4\pi iMn}{k}}, \tag{5.21}$$

and the comparison of this with the analysis of disc two-point function yields,

$$\tilde{\mu}_B \;=\; -i\sin\{\tfrac{\pi}{k}(2J+1)\}\frac{\nu^{\frac{1}{2}}\Gamma(-\frac{1}{k})}{2\pi k}. \tag{5.22}$$

Let us next evaluate the product of OPE coefficients $c^{\updownarrow}\tilde{t}^{\updownarrow}(n,\bar{n})$ They were defined in the previous section as follows,

$$\begin{aligned} V^{k/2}_{n,\bar{n}} \;&\longrightarrow\; \sum_{(+,\uparrow),(-,\downarrow)} \tilde{t}^{\updownarrow}(n,\bar{n})|z-\bar{z}|^{\frac{2n\bar{n}}{k}+\frac{k}{2}+1\mp(n+\bar{n})}B^{\frac{k}{2}(\pm1)}_{n+\bar{n}\mp\frac{k}{2}\mp1} + \cdots, \\ \tilde{t}^{\updownarrow}(n,\bar{n})B^{k/2(\pm1)}_{(n+\bar{n})\mp\frac{k}{2}\mp1} \;&\stackrel{n+\bar{n}\to0}{\longrightarrow}\; c^{\updownarrow}\tilde{t}^{\updownarrow}(n,-n)\Gamma(\pm(n+\bar{n}))\cdot\mathbf{1}. \end{aligned} \tag{5.23}$$

Although we were not aware in the previous section, $t^{\updownarrow}$ are linear in the boundary fermions and the product $c^{\updownarrow}\tilde{t}^{\updownarrow}(n,-n)$ involves the algebra of boundary fermions. Calculating them as follows,

$$\begin{aligned} \tilde{t}^{\uparrow}(n,\bar{n}) &\sim \langle B^{-k/2-1(-1)}_{k/2+1-n-\bar{n}}V^{k/2}_{n,\bar{n}}(-\mu_BS_B-\mu_{\bar{B}}S_{\bar{B}})\rangle_{\rm free} \\ &\sim \{\mu_B\lambda + (-)^{n-\bar{n}}\mu_{\bar{B}}\bar{\lambda}\}\frac{2\pi\Gamma(-k-1-n-\bar{n})}{\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}+\bar{n})}e^{\frac{2\pi iM(n-\bar{n})}{k}}, \\ (a\lambda+b\bar{\lambda})B^{\frac{k}{2}(1)}_{m-\frac{k}{2}-1}(0) &\sim -(\bar{\mu}_B\bar{S}_B+\bar{\mu}_{\bar{B}}\bar{S}_{\bar{B}})(a\lambda+b\bar{\lambda})B^{\frac{k}{2}(1)}_{m-\frac{k}{2}-1}(0) \\ &\sim (a\bar{\mu}_{\bar{B}}+b\bar{\mu}_B)\int_0^{\Lambda}x^{m-1} \;\sim\; (a\bar{\mu}_{\bar{B}}+b\bar{\mu}_B)\Gamma(m), \end{aligned} \tag{5.24}$$

we find

$$c^{\uparrow}t^{\uparrow}(n,-n) \;=\; (\mu_B\bar{\mu}_{\bar{B}}+(-)^{2n}\mu_{\bar{B}}\bar{\mu}_B)\frac{2\pi\Gamma(-k-1)}{\Gamma(-\frac{k}{2}+n)\Gamma(-\frac{k}{2}-n)}e^{\frac{4\pi iMn}{k}}. \tag{5.25}$$

By comparing this with (4.55) we obtain

$$\mu_B \bar{\mu}_{\bar{B}} \;=\; -\frac{k\mu}{2\pi} e^{2\pi i J}, \quad \mu_{\bar{B}} \bar{\mu}_B \;=\; -\frac{k\mu}{2\pi} e^{-2\pi i J}. \tag{5.26}$$

Combining this with (5.10) we can determine the boundary couplings up to a single phase. In the next section we set the couplings as follows,

$$(\mu_B, \mu_{\bar{B}}, \bar{\mu}_B, \bar{\mu}_{\bar{B}}) \;=\; i\sqrt{\frac{k\mu}{2\pi}}(e^{i\pi(J-M)}, e^{-i\pi(J-M)}, e^{-i\pi(J+M)}, e^{i\pi(J+M)}) \tag{5.27}$$

and compute the reflection coefficients of boundary operators. We will read off the open string spectrum from it and find a precise agreement with the result of modular bootstrap of annulus amplitudes.

5.2 A-branes

For A-branes in LG models, it is not known how to construct boundary interactions. However, in the framework of perturbed free CFT, nothing seems to prevent us from incorporating the boundary screening operators of the same form. In this and the following sections we will try to reproduce some disc structure constants involving A-branes using the following boundary action,

$$\begin{aligned}
&\oint dx \left[\bar{\lambda}\partial_x\lambda - \sqrt{\tfrac{2}{k}}\tfrac{K\rho}{4\pi}\right] + \mu_A S_A + \bar{\mu}_A \bar{S}_A + \tilde{\mu}_A \tilde{S}_A \\
&= \oint dx \left[\bar{\lambda}\partial_x\lambda - \sqrt{\tfrac{2}{k}}\tfrac{K\rho}{4\pi} - \mu_A \lambda e^{-\sqrt{\frac{k}{2}}\phi_L + iH_L} - \bar{\mu}_A e^{-\sqrt{\frac{k}{2}}\bar{\phi}_L - iH_L}\bar{\lambda}\right. \\
&\qquad\qquad \left. -\tilde{\mu}_A(\lambda\bar{\lambda} - \bar{\lambda}\lambda)(\psi_+\psi_- - i\sqrt{2k}\partial\theta)e^{-\sqrt{\frac{2}{k}}\rho_L}\right].
\end{aligned} \tag{5.28}$$

An important difference between A- and B-type boundaries in performing free CFT computation is that the boundary condition on free fields θ, H is Dirichlet for A-type and Neumann for B-type. In the same sense, we should regard the boundary interactions as carrying nonzero winding number or momentum for A- or B-type boundaries, respectively.

As we did in the case of B-branes, we regard the system as that of free fields perturbed by bulk and boundary screening operators and compute various disc correlators perturbatively. The correlation between left and right-moving sectors is given by the following set of propagators,

$$\langle \rho_L(z)\rho_R(\bar{z}')\rangle \;=\; -\langle \theta_L(z)\theta_R(\bar{z}')\rangle \;=\; -\langle H_L(z)H_R(\bar{z}')\rangle \;=\; -\ln(z-\bar{z}') + i\pi. \tag{5.29}$$

The bulk operator $V^{j(s,\bar{s})}_{m,\bar{m}}$ appearing in a correlator $\langle\cdots\rangle$ between two boundary operators connected by A-type boundary state $A^{\alpha}_{[J,M]}$ yields a factor

$$\exp\left(\frac{2\pi i}{k}(M+\alpha)(m+\bar{m}+s+\bar{s}) + i\pi\alpha(s+\bar{s})\right)\{e^{\frac{i\pi}{2}(m+\bar{m})}\lambda\bar{\lambda} + e^{-\frac{i\pi}{2}(m+\bar{m})}\bar{\lambda}\lambda\}. \tag{5.30}$$

In order for the bulk and boundary operators to (anti-)commute in accordance with their Grassmann parity, we require the physical boundary operator $B_m^{j(s)}$ between A-branes $A^{\alpha}_{[JM]}$ and $A^{\alpha'}_{[J'M']}$ to satisfy

$$\begin{aligned}
\lambda\bar{\lambda}B_m^{j(s)}, \bar{\lambda}\lambda B_m^{j(s)} &\cdots m \in M - M' + \mathbb{Z}, \qquad s \in \alpha - \alpha' + \mathbb{Z}, \\
\lambda B_m^{j(s)} &\cdots m \in M - M' - \tfrac{k}{2} + \mathbb{Z}, \; s \in \alpha - \alpha' + \mathbb{Z}, \\
\bar{\lambda} B_m^{j(s)} &\cdots m \in M - M' + \tfrac{k}{2} + \mathbb{Z}, \; s \in \alpha - \alpha' + \mathbb{Z}.
\end{aligned} \tag{5.31}$$

The fermion number of boundary operator is given by $s - \alpha + \alpha' - \sharp(\lambda) + \sharp(\bar{\lambda})$ and is always an integer. In the next section we will discuss a little more about the above condition for the physical spectrum.

One can relate the boundary couplings $(\mu_A, \bar{\mu}_A, \tilde{\mu}_A)$ with the labels $[J, M]$ of A-branes by computing the coefficients $u(n)$, $\tilde{u}^{\updownarrow}(n, \bar{n})$ and $c^{\updownarrow}$ perturbatively and comparing the results with those in the previous section. One finds

$$\begin{aligned}
\tilde{\mu}_A &= -\cos\{\tfrac{\pi}{k}(2J+1)\}\frac{\nu^{\frac{1}{2}}\Gamma(-\frac{1}{k})}{2\pi k}, \\
\mu_A\bar{\mu}_A &= \frac{2k\mu}{\pi}\sin\pi(J+M)\sin\pi(J-M).
\end{aligned} \tag{5.32}$$

In the next section we compute the reflection coefficients of boundary operators using the values of $(\mu_A, \bar{\mu}_A)$

$$\mu_A = \sqrt{\frac{2k\mu}{\pi}}\sin\pi(J-M), \qquad \bar{\mu}_A = \sqrt{\frac{2k\mu}{\pi}}\sin\pi(J+M). \tag{5.33}$$

Let us finally point out that the bulk-boundary propagator $\langle V^{k/2}_{n,\pm n} B_0^{-1}\rangle$ ($\pm$ signs correspond to A- and B-branes respectively) exactly vanishes when evaluated as a screening integral. This is in consistency with the observation of the previous section that the self-OPE of degenerate operator $V^{k/2}_{n,\pm n}$ does not yield boundary identity operator in a simple manner.

6. Boundary reflection coefficients

Now that the wave functions for various boundary states are available, one can obtain the spectrum of open strings between any branes from the modular transformation property of annulus amplitudes. For example, the annulus amplitude between two A-branes both corresponding to non-chiral non-degenerate representations is calculated as

$$\begin{aligned}
Z &= \langle A^{\alpha,\beta}_{[J,M]}| e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})} | A^{\alpha',-\beta}_{[J',M']}\rangle \\
&= \sum_{m+\beta\in\frac{k}{2}\mathbb{Z}} \int \frac{dj}{i\pi} U_{[J,M]}(-j-1,-m,-\beta) U_{[J',M']}(j,m,\beta)\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha).
\end{aligned} \tag{6.1}$$

Rewriting this as a sum of characters in the open string channel we obtain

$$Z = \sum_{n\in\mathbb{Z}+\alpha-\alpha'} e^{2\pi i\beta(\alpha'-\alpha-\frac{2}{k}(n+M-M'))}\int_{-\infty}^{\infty} ds\,\Big\{\rho_0^A(s|J,J')\chi_{-\frac{1}{2}+is,n+M-M',n}(\tau_o,\beta)$$
$$+\sum_{\pm}\rho_1^A(s|J,J')\chi_{-\frac{1}{2}+is,n+M-M'\pm\frac{k}{2},n}(\tau_o,\beta)\Big\}\,, \quad (6.2)$$

with

$$\rho_0^A(s|J,J') = \int_{-\infty}^{\infty} dp e^{2\pi ips}\frac{\cosh\{\pi p(2J+1)\}\cosh\{\pi p(2J'+1)\}\cosh\{k\pi p\}}{\sinh(\pi p)\sinh(\pi kp)},$$
$$\rho_1^A(s|J,J') = \int_{-\infty}^{\infty} dp e^{2\pi ips}\frac{\cosh\{\pi p(2J+1)\}\cosh\{\pi p(2J'+1)\}}{2\sinh(\pi p)\sinh(\pi kp)}. \quad (6.3)$$

As a non-trivial consistency check, here we try to read off the spectrum from a different approach using the reflection coefficients of boundary operators. This also enables one to find and check the correspondences between various form of boundary interactions and the boundary states. We will heavily apply the techniques developed in $N = 0$ and $N = 1$ Liouville theories[8, 12] that use boundary degenerate operators.

Let us take the upper half plane with A- or B-type boundary conditions. The boundary operators are labelled by the two D-branes they are ending on, as well as their Chan-Paton indices. Note that each single D-brane is defined with two-dimensional Chan-Paton space. So the boundary operators are 2×2 matrices. We denote the four matrix elements of boundary operators as

$$[\lambda\bar{\lambda}B_m^{l(s)}]^X_{X'},\quad [\bar{\lambda}\lambda B_m^{l(s)}]^X_{X'},\quad [\lambda B_m^{l(s)}]^X_{X'},\quad [\bar{\lambda}B_m^{l(s)}]^X_{X'} \quad (6.4)$$

where X and X' are the sets of parameters specifying the boundary states appearing on its left and right:

$$X = \{[J,M];\alpha\},\quad X' = \{[J',M'];\alpha'\}. \quad (6.5)$$

α labels the rotation by R-symmetry.

In this section we order the boundary operators in OPE formulae or correlators as they appear on the real axis. In computing correlators using the prescription of previous section, we need to re-order the operators first and then Wick contract. Note that this also involves the re-ordering of composite operators like $\lambda\bar{\lambda}B_m^{l(s)}$.

6.1 A-branes

In the previous section we have argued that the boundary operator $[B_m^{l(s)}]^X_{X'}$ can connect the two boundary states $X = \{[J,M],\alpha\}$ and $X' = \{[J',M'],\alpha'\}$ only when it satisfies a certain quantization condition. From the annulus amplitude (6.2) one finds

$$s \in \alpha-\alpha'+\mathbb{Z},\quad m \in M-M'+\mathbb{Z}\ \text{ or } M-M'\pm\tfrac{k}{2}+\mathbb{Z}. \quad (6.6)$$

Recall that the A-branes in $N = 2$ Liouville theory are extending orthogonally to the periodic direction and M can be understood as their position. Boundary operators therefore carry winding number m, and the above condition says it can differ from naive values $M - M' + \mathbb{Z}$ by $\pm\frac{k}{2}$. This mild breaking of winding number quantization law should be understood as due to boundary interaction terms. Moreover, the mild-ness of the winding number violation should be due to the boundary fermions. Naively, if there were boundary operators with $m = M - M' + \frac{k}{2}$, one would get operators with $m = M - M' + \frac{kN}{2}$ $(N \geq 2)$ by fusing them. However, the operators with $N \geq 2$ never appear if we require

$$\begin{aligned} &m \in M - M' + \mathbb{Z} \qquad \text{for } [\lambda\bar{\lambda}B_m^{l(s)}]^X_{\ X'},\ [\bar{\lambda}\lambda B_m^{l(s)}]^X_{\ X'}, \\ &m \in M - M' + \tfrac{k}{2} + \mathbb{Z} \text{ for } [\bar{\lambda}B_m^{l(s)}]^X_{\ X'}, \\ &m \in M - M' - \tfrac{k}{2} + \mathbb{Z} \text{ for } [\lambda B_m^{l(s)}]^X_{\ X'}, \end{aligned} \tag{6.7}$$

simply because $(\bar{\lambda})^N$ vanishes if $N \geq 2$. This is consistent with the physical condition (5.31) we found from locality in the free field picture.

As a reflection coefficient of boundary operators, we first consider

$$[B_m^{l(s)}]^X_{\ X'} = d(l, m, s; X; X')[B_m^{-l-1(s)}]^X_{\ X'}. \tag{6.8}$$

Let us first find out what kind of matrix the coefficient d is. Recall that the reflection coefficients are related to the two-point functions:

$$d(l, m, s; X; X') \ \sim\ \langle [B_m^{l(s)}]^X_{\ X'} [B_{-m}^{l(-s)}]^{X'}_{\ X} \rangle. \tag{6.9}$$

From the quantization law of m it follows that the boundary operator $\lambda\bar{\lambda}B$ has nonzero two-point functions with $\lambda\bar{\lambda}B$ or $\bar{\lambda}\lambda B$, and λB has nonzero two-point functions only with $\bar{\lambda}B$. For the operators $\lambda\bar{\lambda}B, \bar{\lambda}\lambda B$ the reflection coefficient therefore becomes a 2×2 matrix:

$$\begin{pmatrix} \lambda\bar{\lambda}B_m^{l(s)} \\ \bar{\lambda}\lambda B_m^{l(s)} \end{pmatrix} = \begin{pmatrix} d^{\lambda\bar{\lambda}}_{\ \lambda\bar{\lambda}} & d^{\lambda\bar{\lambda}}_{\ \bar{\lambda}\lambda} \\ d^{\bar{\lambda}\lambda}_{\ \lambda\bar{\lambda}} & d^{\bar{\lambda}\lambda}_{\ \bar{\lambda}\lambda} \end{pmatrix} \begin{pmatrix} \lambda\bar{\lambda}B_m^{-l-1(s)} \\ \bar{\lambda}\lambda B_m^{-l-1(s)} \end{pmatrix}. \tag{6.10}$$

For the operators $\lambda B, \bar{\lambda}B$ the reflection coefficients will be ordinary numbers. Finding these coefficients is our primary goal in the following arguments.

To begin with, let us consider the 2×2 matrix-valued reflection coefficient $d(l, m, s; X; X')$ for boundary operators $\lambda\bar{\lambda}B, \bar{\lambda}\lambda B$. First of all, it follows from the quantization condition on m and s that

$$d(l, m, s; X; X') \sim \delta^{(1)}_{m, M-M'}\delta^{(1)}_{s, \alpha-\alpha'} \tag{6.11}$$

To derive further constraints on d, we have to analyze the boundary OPEs involving degenerate operators. Consider the following OPEs

$$\begin{aligned} &\begin{pmatrix} [\lambda\bar{\lambda}B_m^{l(s)}]^X_{\ X'} \\ [\bar{\lambda}\lambda B_m^{l(s)}]^X_{\ X'} \end{pmatrix} \times\ [B_{m_0}^{k/2}]^{X'}_{\ X''} \longrightarrow \tilde{c}^R_+ \begin{pmatrix} [\lambda\bar{\lambda}B_{m+m_0}^{l+k/2(s)}]^X_{\ X''} \\ [\bar{\lambda}\lambda B_{m+m_0}^{l+k/2(s)}]^X_{\ X''} \end{pmatrix} + \tilde{c}^R_- \begin{pmatrix} [\lambda\bar{\lambda}B_{m+m_0}^{l-k/2(s)}]^X_{\ X''} \\ [\bar{\lambda}\lambda B_{m+m_0}^{l-k/2(s)}]^X_{\ X''} \end{pmatrix} \\ &[B_{m_0}^{k/2}]^{X''}_{\ X} \times \begin{pmatrix} [\lambda\bar{\lambda}B_m^{l(s)}]^X_{\ X'} \\ [\bar{\lambda}\lambda B_m^{l(s)}]^X_{\ X'} \end{pmatrix} \longrightarrow \tilde{c}^L_+ \begin{pmatrix} [\lambda\bar{\lambda}B_{m+m_0}^{l+k/2(s)}]^{X''}_{\ X'} \\ [\bar{\lambda}\lambda B_{m+m_0}^{l+k/2(s)}]^{X''}_{\ X'} \end{pmatrix} + \tilde{c}^L_- \begin{pmatrix} [\lambda\bar{\lambda}B_{m+m_0}^{l-k/2(s)}]^{X''}_{\ X'} \\ [\bar{\lambda}\lambda B_{m+m_0}^{l-k/2(s)}]^{X''}_{\ X'} \end{pmatrix} \end{aligned} \tag{6.12}$$

where X's are abbreviations for the labels of branes,

$$X = [JM\alpha], \quad X' = [J'M'\alpha'], \quad X'' = [J''M''\alpha''] \tag{6.13}$$

and $m \in M - M' + \mathbb{Z},\ m_0 = M' - M'',\ \alpha' - \alpha'' = 0$ in the first line and similarly for the second line. The coefficients $\tilde{c}_\pm^{R,L}$ are 2×2 matrices in the same way as d. One finds as usual $\tilde{c}_+^{R,L} \equiv 1$, and $\tilde{c}_-^{R,L}$ give a set of recursion relations for the reflection coefficient. For example, by considering the term proportional to $B_{m+m_0}^{-l-1+\frac{k}{2}(s)}$ in the product of $B_m^{l(s)}$ and $B_{m_0}^{k/2}$ one obtains

$$\begin{aligned} d(l,m,s;X;X') &= \tilde{c}_-^R(l,m,s,m_0;X;X';X'')d(l-\tfrac{k}{2},m+m_0,s;X;X''), \\ d(l,m,s;X;X') &= \tilde{c}_-^L(l,m,s,m_0;X'';X;X')d(l-\tfrac{k}{2},m+m_0,s;X'';X'). \end{aligned} \tag{6.14}$$

Another set of recursion relations follows by considering the term proportional to $B_{m+m_0}^{-l-1-\frac{k}{2}(s)}$,

$$\begin{aligned} d(l+\tfrac{k}{2},m+m_0,s;X;X'') &= d(l,m,s;X;X')\tilde{c}_-^R(-l-1,m,s,m_0;X;X';X''), \\ d(l+\tfrac{k}{2},m+m_0,s;X'';X') &= d(l,m,s;X;X')\tilde{c}_-^L(-l-1,m,s,m_0;X'';X;X'), \end{aligned} \tag{6.15}$$

but the former two are related to the latter two due to

$$d(l,m,s;X;X')d(-l-1,m,s;X;X') = 1. \tag{6.16}$$

Also, the two equations in (6.14) are not independent once we notice

$$\begin{aligned} d(l,m,s;X;X')^t &= d(l,-m,-s;X';X), \\ \tilde{c}_-^L(l,m,s,m_0;X'';X;X')^t &= \tilde{c}_-^R(-l-1+\tfrac{k}{2},-m-m_0,-s,m_0;X';X'';X), \end{aligned} \tag{6.17}$$

from their relation to disc correlators.

The matrix coefficients $\tilde{c}_-^{L,R}$ are calculated as screening integrals:

$$\begin{aligned} \tilde{c}_-^L(l,m,s,m_0;X;X';X'') &= \langle B_m^{l(s)} B_{m_0}^{k/2} B_{-m-m_0}^{-l-1+k/2(-s)} \rangle \\ &= \langle B_m^{l(s)} B_{m_0}^{k/2} B_{-m-m_0}^{-l-1+k/2(-s)} (-\mu S - \bar{\mu}\bar{S} + \mu_A S_A \bar{\mu}_A \bar{S}_A) \rangle_{\text{free}}. \end{aligned} \tag{6.18}$$

There are two kinds of contributions to the coefficient $\tilde{c}_-$, one proportional to $\mu_A\bar{\mu}_A$ and the other proportional to μ or $\bar{\mu}$. The first ones are expressed as the following integral

$$I = \int ds d\bar{s} |s|^{l-m} |\bar{s}|^{l+m} |1-s|^{\frac{k}{2}-m_0} |1-\bar{s}|^{\frac{k}{2}+m_0} |s-\bar{s}|^{-k-1}, \tag{6.19}$$

with different integration domains. As before, we consider the three segments of the real line

$$(1)\ [-\infty, 0], \quad (2)\ [0,1], \quad (3)\ [1,\infty]$$

and denote the integrals with suffices indicating the integration domain of $s, \bar{s}$. For example,

$$I_{1\bar{1}} \Leftrightarrow \{s < \bar{s} < 0\}, \quad I_{\bar{1}2} \Leftrightarrow \{\bar{s} < 0 < s < 1\}, \quad I_{2\bar{3}} \Leftrightarrow \{0 < s < 1 < \bar{s}\}, \quad \text{etc.}$$

These integrals can be expressed in terms of the functions G_k defined in the appendix, where some useful formulae are also presented. The integrals $I_{1\bar{1}}$ and $I_{\bar{1}1}$ have simple expressions

$$\begin{aligned} I_{1\bar{1}} &= \frac{\Gamma(1+l+m)\Gamma(-l+\frac{k}{2}+m+m_0)}{\Gamma(-l+m)\Gamma(1+l-\frac{k}{2}+m+m_0)}\Gamma(-2l-1)\Gamma(2l-k+1), \\ I_{\bar{1}1} &= \frac{\Gamma(1+l-m)\Gamma(-l+\frac{k}{2}-m-m_0)}{\Gamma(-l-m)\Gamma(1+l-\frac{k}{2}-m-m_0)}\Gamma(-2l-1)\Gamma(2l-k+1) \end{aligned} \tag{6.20}$$

but others do not. Using them, the matrix elements are computed as

$$\begin{aligned} \tilde{c}^R_-(l,m,s,m_0;X;X';X'')^{\lambda\bar{\lambda}}_{\lambda\bar{\lambda}} &= \mu_A\bar{\mu}_A I_{1\bar{1}} + \mu'_A\bar{\mu}'_A I_{2\bar{2}} + \mu''_A\bar{\mu}''_A I_{3\bar{3}} + \mu'_A\bar{\mu}''_A I_{2\bar{3}} + I_0(\mu,\bar{\mu})^{\lambda\bar{\lambda}}_{\lambda\bar{\lambda}}, \\ \tilde{c}^R_-(l,m,s,m_0;X;X';X'')^{\bar{\lambda}\lambda}_{\bar{\lambda}\lambda} &= \mu_A\bar{\mu}_A I_{\bar{1}1} + \mu'_A\bar{\mu}'_A I_{\bar{2}2} + \mu''_A\bar{\mu}''_A I_{\bar{3}3} + \mu''_A\bar{\mu}'_A I_{\bar{2}3} + I_0(\mu,\bar{\mu})^{\bar{\lambda}\lambda}_{\bar{\lambda}\lambda}, \\ \tilde{c}^R_-(l,m,s,m_0;X;X';X'')^{\lambda\bar{\lambda}}_{\bar{\lambda}\lambda} &= \{\mu'_A\bar{\mu}_A I_{\bar{1}2} + \mu''_A\bar{\mu}_A I_{\bar{1}3}\}(-)^{s-\alpha+\alpha'}, \\ \tilde{c}^R_-(l,m,s,m_0;X;X';X'')^{\bar{\lambda}\lambda}_{\lambda\bar{\lambda}} &= \{\mu_A\bar{\mu}'_A I_{1\bar{2}} + \mu_A\bar{\mu}''_A I_{1\bar{3}}\}(-)^{s-\alpha+\alpha'}. \end{aligned} \tag{6.21}$$

and similarly for $\tilde{c}^L_-$. In this expression, the coupling constants $\mu_A, \bar{\mu}_A$ are functions of (J, M) as given in (5.33) and similarly for those with primes. The diagonal elements of $\tilde{c}^R_-$ also have $\mathcal{O}(\mu,\bar{\mu})$ contribution I_0 which are given by

$$\begin{aligned} (I_0)^{\lambda\bar{\lambda}}_{\lambda\bar{\lambda}} &= -\frac{k\mu}{\pi}\left\{\mathbf{c}(2M-k)I_{\bar{1}1} + \mathbf{c}(2M')I_{2\bar{2}} + \mathbf{c}(2M'')I_{3\bar{3}} + \mathbf{c}(M'+M''-\tfrac{k}{2})I_{2\bar{3}}\right\} \\ (I_0)^{\bar{\lambda}\lambda}_{\bar{\lambda}\lambda} &= -\frac{k\mu}{\pi}\left\{\mathbf{c}(2M+k)I_{1\bar{1}} + \mathbf{c}(2M')I_{\bar{2}2} + \mathbf{c}(2M'')I_{\bar{3}3} + \mathbf{c}(M'+M''+\tfrac{k}{2})I_{\bar{2}3}\right\}. \end{aligned} \tag{6.22}$$

After the $\mu_A, \bar{\mu}_A$ are substituted with the functions (5.33), the expression for $\tilde{c}^{L,R}_-$ simplifies under the following assumption

> *the degenerate operators $[B^{k/2}_m]^X_{X'}$ only appear between branes $X = [J, M, \alpha]$ and $X' = [J', M', \alpha']$ satisfying $J - J' = \frac{k}{2}$.*

The OPE coefficients $\tilde{c}^{L,R}_-$ are then given by

$$\begin{aligned} &\tilde{c}^R_-(l,m,s,m_0;X;X';X'')\big|_{m_0=M'-M'',\ J'=J''+\frac{k}{2},\ \alpha'=\alpha''} \\ &= \frac{2k\mu}{\pi}\begin{pmatrix}\gamma & 0\\ 0 & \bar{\gamma}\end{pmatrix}\begin{pmatrix}\mathbf{s}(J+M) & -\mathbf{s}(J'+l-M)\\ -\mathbf{s}(J'+l+M) & \mathbf{s}(J-M)\end{pmatrix} \\ &\quad\times\begin{pmatrix}I_{1\bar{1}} & 0\\ 0 & I_{\bar{1}1}\end{pmatrix}\begin{pmatrix}\mathbf{s}(J-M) & \mathbf{s}(J'+l-k-M)\\ \mathbf{s}(J'+l-k+M) & \mathbf{s}(J+M)\end{pmatrix}\begin{pmatrix}\bar{\gamma} & 0\\ 0 & \gamma\end{pmatrix}, \\ &\tilde{c}^L_-(l,m,s,m_0;X'';X;X')\big|_{m_0=M''-M,\ J''=J+\frac{k}{2},\ \alpha=\alpha''} \\ &= \frac{2k\mu}{\pi}\begin{pmatrix}\gamma & 0\\ 0 & \bar{\gamma}\end{pmatrix}\begin{pmatrix}\mathbf{s}(J'-M') & -\mathbf{s}(J-l+M')\\ -\mathbf{s}(J-l-M') & \mathbf{s}(J'+M')\end{pmatrix} \\ &\quad\times\begin{pmatrix}I_{\bar{1}1} & 0\\ 0 & I_{1\bar{1}}\end{pmatrix}\begin{pmatrix}\mathbf{s}(J'+M') & \mathbf{s}(J-l+k+M')\\ \mathbf{s}(J-l+k-M') & \mathbf{s}(J'-M')\end{pmatrix}\begin{pmatrix}\bar{\gamma} & 0\\ 0 & \gamma\end{pmatrix}, \\ &\gamma = \exp\left\{\tfrac{i\pi}{2}(M-M'-m+\alpha-\alpha'-s)\right\} \end{aligned} \tag{6.23}$$

where $I_{1\bar{1}}, I_{\bar{1}1}$ are as given in (6.20).

It turns out very non-trivial to check that the set of recursion relations are consistent (solvable) and has a solution with the appropriate symmetry properties (6.16),(6.17). We find that the solution can be written in terms of the special functions $\mathbf{G}$ and $\mathbf{S}$ introduced in [8] (see the appendix for their definitions) as follows:

$$\begin{aligned} d(l,m,s;X;X') &= (\nu b^{2-2b^2})^{l+\frac{1}{2}} \frac{\mathbf{G}(b(2l+1))}{\mathbf{G}(-b(2l+1))} \mathbf{S}(b(l+J+J'+2)) \mathbf{S}(b(l+J-J'+1)) \\ &\quad \times \mathbf{S}(b(l-J+J'+1)) \mathbf{S}(b(l-J-J')) \times \hat{d}(l,m,s;X;X'), \end{aligned} \tag{6.24}$$

where $b \equiv k^{-1/2}$ as before. The matrix part $\hat{d}(l,m,s;X;X')$ is given by

$$\begin{aligned} \hat{d}(l,m,s;X;X') &= 4 \begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{s}(J'-M') & -\mathbf{s}(J-l+M') \\ -\mathbf{s}(J-l-M') & \mathbf{s}(J'+M') \end{pmatrix} \begin{pmatrix} \frac{\Gamma(1+l-m)}{\Gamma(-l-m)} & 0 \\ 0 & \frac{\Gamma(1+l+m)}{\Gamma(-l+m)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \mathbf{s}(J'+M') & -\mathbf{s}(J+l+M') \\ -\mathbf{s}(J+l-M') & \mathbf{s}(J'-M') \end{pmatrix} \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} \\ &= 4 \begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{s}(J+M) & -\mathbf{s}(J'+l-M) \\ -\mathbf{s}(J'+l+M) & \mathbf{s}(J-M) \end{pmatrix} \begin{pmatrix} \frac{\Gamma(1+l+m)}{\Gamma(-l+m)} & 0 \\ 0 & \frac{\Gamma(1+l-m)}{\Gamma(-l-m)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \mathbf{s}(J-M) & -\mathbf{s}(J'-l-M) \\ -\mathbf{s}(J'-l+M) & \mathbf{s}(J+M) \end{pmatrix} \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix}, \\ \gamma &= \exp\left\{ \tfrac{i\pi}{2}(M-M'-m+\alpha-\alpha'-s) \right\} \end{aligned} \tag{6.25}$$

We can solve similar equations for reflection coefficients for operators $[\lambda B_m^{l(s)}]$ or $[\bar{\lambda}\, B_m^{l(s)}]$. The reflection coefficients $d_\lambda, d_{\bar{\lambda}}$ for these operators are ordinary numbers, so one expects them to be proportional to $\frac{\Gamma(1+l\pm m)}{\Gamma(-l\pm m)}$. Then $\tilde{c}_-^{L,R}$ should be proportional to either $I_{1\bar{1}}$ or $I_{\bar{1}1}$ and not their linear combination. Let us consider the case of $[\lambda B_m^{l(s)}]$ and calculate the OPE coefficients $\tilde{c}_-^{L,R}$ as screening integrals. We find that the sum of bulk and boundary screening integrals takes the simple form:

$$\begin{aligned} \tilde{c}_-^R(l,m,s,m_0;X;X',X'') &= -\frac{2k\mu}{\pi} \mathbf{s}(l+J+J'-\tfrac{k}{2}) \mathbf{s}(l-J+J'-\tfrac{k}{2}) I_{1\bar{1}}, \\ \tilde{c}_-^L(l,m,s,m_0;X;X',X'') &= -\frac{2k\mu}{\pi} \mathbf{s}(l-J-J'-\tfrac{k}{2}) \mathbf{s}(l-J+J'-\tfrac{k}{2}) I_{1\bar{1}}. \end{aligned} \tag{6.26}$$

where we imposed again $m_0 = M'-M''$, $J' = J''+\frac{k}{2}$, $\alpha' = \alpha''$ in the first line and similarly in the second line, too. The reflection coefficient $d_\lambda(l,m;X;X')$ is obtained as before by solving a set of recursion relations. One finds

$$\begin{aligned} d_\lambda(l,m;X;X') &= (\nu b^{2-2b^2})^{l+\frac{1}{2}} \frac{\Gamma(1+l+m)}{\Gamma(-l+m)} \frac{\mathbf{G}(b(2l+1))}{\mathbf{G}(-b(2l+1))} \\ &\quad \times \mathbf{S}(b(l+J+J'+2+\tfrac{k}{2})) \mathbf{S}(b(l+J-J'+1+\tfrac{k}{2})) \\ &\quad \times \mathbf{S}(b(l-J+J'+1+\tfrac{k}{2})) \mathbf{S}(b(l-J-J'+\tfrac{k}{2})). \end{aligned} \tag{6.27}$$

Boundary chiral operators are expected to satisfy a different kind of reflection relation, which should be of the form

$$[\bar{\lambda}B^{l(s)}_{\mp l}]^{X}_{\ X'} \;=\; d^{\pm}(l,s;X;X')\begin{pmatrix}[\lambda\bar{\lambda}B^{\tilde{l}(s+1)}_{\pm\tilde{l}}]^{X}_{\ X'}\\ [\bar{\lambda}\lambda B^{\tilde{l}(s+1)}_{\pm\tilde{l}}]^{X}_{\ X'}\end{pmatrix}, \tag{6.28}$$

where $\tilde{l}=-l-1-\frac{k}{2}$. Here $d^{\pm}$ is a two-component row vector with components $(d^{\pm}_{\lambda\bar{\lambda}},d^{\pm}_{\bar{\lambda}\lambda})$. Once we know the OPE coefficient $\tilde{c}^R_{\downarrow}=((\tilde{c}^R_{\downarrow})_{\lambda\bar{\lambda}},(\tilde{c}^R_{\downarrow})_{\bar{\lambda}\lambda})$ defined by

$$[\lambda B^{l(s)}_{l}]^{X}_{\ X'}[B^{k/2}_{m}]^{X'}_{\ X''}\longrightarrow((\tilde{c}^R_{\downarrow})_{\lambda\bar{\lambda}},\ (\tilde{c}^R_{\downarrow})_{\bar{\lambda}\lambda})\begin{pmatrix}[\lambda\bar{\lambda}B^{l(s-1)}_{l+m+1+k/2}]^{X}_{\ X''}\\ [\bar{\lambda}\lambda B^{l(s-1)}_{l+m+1+k/2}]^{X}_{\ X''}\end{pmatrix}+\cdots, \tag{6.29}$$

the reflection coefficient d^- can be calculated as

$$\begin{aligned}d^{-}(l,s;X;X') &= \tilde{c}^R_{\downarrow}(l,m,s;X;X';X'')d(l,l+m+\tfrac{k}{2}+1,s-1;X;X''),\\ &\qquad (m\;=\;M'-M'',\quad J'-J''\;=\;\tfrac{k}{2},\quad \alpha'=\alpha''),\end{aligned} \tag{6.30}$$

and similarly for the other one. After some computation one obtains

$$\begin{aligned}(c^R_{\downarrow})(l,m,s;X;X';X'') &= -\sqrt{\frac{2k\mu}{\pi}}\frac{\Gamma(2l+1)\Gamma(-2l-1-m-\frac{k}{2})}{\Gamma(-m-\frac{k}{2})}\\ &\qquad\times\Big(\mathbf{s}(J'+M'+2l)(-)^{\alpha-\alpha'-s},\ \mathbf{s}(J+M)\Big),\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}d^{\pm}(l,s;X;X') &= \pm 2(\nu b^{-2b^2})^{l+\frac{1}{2}+\frac{k}{4}}\frac{\mathbf{G}(b(2l+k+1))}{\mathbf{G}(-b(2l+1))}\\ &\quad\times\mathbf{S}(b(l+J+J'+2+\tfrac{k}{2}))\mathbf{S}(b(l+J-J'+1+\tfrac{k}{2}))\\ &\quad\times\mathbf{S}(b(l-J+J'+1+\tfrac{k}{2}))\mathbf{S}(b(l-J-J'+\tfrac{k}{2}))\times\hat{d}^{\pm}(l,s;X;X'),\\ \hat{d}^{+}(l,s;X;X') &= \Big(\mathbf{s}(J-M),\ (-)^{\alpha-\alpha'-s-1}\mathbf{s}(J'-M')\Big),\\ \hat{d}^{-}(l,s;X;X') &= \Big((-)^{\alpha-\alpha'-s-1}\mathbf{s}(J'+M'),\ \mathbf{s}(J+M)\Big).\end{aligned} \tag{6.32}$$

There are some consistency checks we can do. As an example, one can consider another important boundary OPE involving $l=1/2$ degenerate operators

$$\begin{aligned}[B^{l}_{m}]^{X}_{\ X'}[B^{1/2}_{m_0}]^{X'}_{\ X''} &\longrightarrow c^R_{+}[B^{l+1/2}_{m+m_0}]^{X}_{\ X''}+c^R_{-}[B^{l-1/2}_{m+m_0}]^{X}_{\ X''},\\ [B^{1/2}_{m_0}]^{X''}_{\ X}[B^{l}_{m}]^{X}_{\ X'} &\longrightarrow c^L_{+}[B^{l+1/2}_{m+m_0}]^{X''}_{\ X'}+c^L_{-}[B^{l-1/2}_{m+m_0}]^{X''}_{\ X'},\end{aligned} \tag{6.33}$$

and calculate $c^{R,L}_{-}$ as screening integrals which are proportional to $\tilde{\mu}_A$. On the other hand, they are also calculated as certain ratios of the reflection coefficients obtained above. Comparing the two results we obtain

$$\tilde{\mu}_A \;=\; -\,\nu^{\frac{1}{2}}\frac{\Gamma(-\frac{1}{k})}{2k\pi}\cos\tfrac{\pi}{k}(2J+1), \tag{6.34}$$

in consistency with the result of $\tilde{\mu}_A$ of the previous section (5.22).

Finally, let us try reading off the open string spectrum from the reflection coefficients and matching with the result of modular bootstrap analysis (6.2). The reflection coefficients are essentially the phase shifts of wave functions that are scattered off the Liouville wall, so by taking its log derivative with respect to the Liouville momentum (l quantum number) one should be able to read off the spectrum density. For $d(l,m,s;X;X')$ which is a matrix-valued quantity, it is natural to define the phase shift by the logarithm of its determinant. Discarding the factors which are independent of J, J' and irrelevant one obtains

$$\begin{aligned}\log\det d(l,m,s;X;X') \sim\ & \log[\mathbf{S}(b(l+J+J'+2))\mathbf{S}(b(l+J+J'+2+k)) \\ & \times\mathbf{S}(b(l+J-J'+1))\mathbf{S}(b(l+J-J'+1+k)) \\ & \times\mathbf{S}(b(l-J+J'+1))\mathbf{S}(b(l-J+J'+1+k)) \\ & \times\mathbf{S}(b(l-J-J'))\mathbf{S}(b(l-J-J'+k))] \qquad (6.35) \\ =\ & -2\int_{-\infty}^{\infty}\frac{dp}{p}\frac{e^{(2l+1)\pi p}\cosh\{(2J+1)\pi p\}\cosh\{(2J'+1)\pi p\}\cosh\{k\pi p\}}{\sinh(\pi p)\sinh(k\pi p)}.\end{aligned}$$

The l-derivative of this agrees with the spectral function ρ_0^A in (6.2) up to numerical factors. Similarly, the logarithm of $d_\lambda(l,m,s;X;X')$ is given by

$$\log d_\lambda(l,m,s;X;X') \sim -\int_{-\infty}^{\infty}\frac{dp}{p}\frac{e^{(2l+1)\pi p}\cosh\{(2J+1)\pi p\}\cosh\{(2J'+1)\pi p\}}{\sinh(\pi p)\sinh(k\pi p)}, \qquad (6.36)$$

and its l-derivative agrees with ρ_1^A in (6.2) up to numerical factors.

6.2 B-brane

Using the wave functions for B-branes one can compute the open string spectrum between two B-branes $X=\{[J,M],\alpha\}$ and $X'=\{[J',M'],\alpha'\}$:

$$\begin{aligned}Z &= \langle B^{\alpha,\beta}_{[J,M]}|e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}|B^{\alpha',-\beta}_{[J',M']}\rangle \\ &= \sum_{m\in\frac{1}{2}\mathbb{Z}}\int_{\mathcal{C}_0}\frac{dj}{i\pi}T_{[J,M]}(-j-1,-m,-\beta)T_{[J',M']}(j,m,\beta)\chi_{j,m+\beta,\beta}(\tau_c,\alpha'-\alpha) \\ &= \frac{2kT_0^2}{\pi}e^{-2\pi i\beta(\alpha-\alpha'+\frac{2}{k}(\alpha-\alpha'+M-M'))}\sum_{m\in k\mathbb{Z}+M-M'+\alpha-\alpha'} \qquad (6.37) \\ &\quad\int ds\left\{\chi_{-\frac{1}{2}+is,m,\alpha-\alpha'}(\tau_o,\beta)\rho_0^B(s|J,J')+\chi_{-\frac{1}{2}+is,m+\frac{k}{2},\alpha-\alpha'}(\tau_o,\beta)\rho_1^B(s|J,J')\right\},\end{aligned}$$

where $T_{[J,M]}(j,m,\beta)$ is the wave function of (4.55) and T_0 is its normalization which is so far undetermined. The spectral functions $\rho^B_{0,1}$ are given by

$$\begin{aligned}\rho_0^B(s|J,J') &= \int dp\frac{e^{2\pi ips}\left[\cosh\{2\pi p(J-J')\}\cosh(k\pi p)+\cosh\{2\pi p(J+J'+1)\}\right]}{\sinh(\pi p)\sinh(k\pi p)}, \\ \rho_1^B(s|J,J') &= \int dp\frac{e^{2\pi ips}\left[\cosh\{2\pi p(J+J'+1)\}\cosh(k\pi p)+\cosh\{2\pi p(J-J')\}\right]}{\sinh(\pi p)\sinh(k\pi p)}. \qquad (6.38)\end{aligned}$$

JHEP12(2006)061

We would like to reproduce this from the boundary reflection coefficients. Notice first of all that the condition on physical open string operators (5.18) is in accordance with the spectrum that can be read off from the annulus amplitude. Let us recapitulate it here:

$$\begin{aligned}
&s\in\alpha-\alpha'+\mathbb{Z}, \quad m+s\in\alpha-\alpha'-M+M'+k\mathbb{Z} \qquad \text{for } [\lambda\bar\lambda B_m^{l(s)}]^X_{X'},\ [\bar\lambda\lambda B_m^{l(s)}]^X_{X'},\\
&s\in\alpha-\alpha'+\mathbb{Z}, \quad m+s\in\alpha-\alpha'-M+M'+k\mathbb{Z}+\tfrac{k}{2} \ \text{for } [\lambda B_m^{l(s)}]^X_{X'},\ [\bar\lambda B_m^{l(s)}]^X_{X'}.
\end{aligned} \tag{6.39}$$

We first consider the 2×2 matrix-valued reflection coefficient $d(l,m,s;X;X')$ for the operators $(\lambda\bar\lambda B_m^{l(s)},\ \bar\lambda\lambda B_m^{l(s)})$, which is defined in the same way as for A-branes. To obtain it, we analyze the recursion relations arising from the boundary OPEs involving $l=k/2$ degenerate operators. Calculation of the OPE coefficients $\tilde c_-^{L,R}$ goes in a similar way as before, except that the bulk screening operators do not show up. Under the assumption that the degenerate operator $[B_m^{k/2}]^X_{X'}$ connects two boundary states only when $J=J'+\frac{k}{2}$, we obtain

$$\begin{aligned}
&\tilde c_-^R(l,m,s,m_0;X;X';X'')|_{m_0=M'-M'',\ J'=J''+\frac{k}{2},\ \alpha'=\alpha''}\\
&=-\frac{ik\mu}{\pi}\mathbf{s}(l+J+J'-\tfrac{k}{2})\begin{pmatrix}\gamma & 0\\ 0 & \bar\gamma\end{pmatrix}\begin{pmatrix}\xi & -\bar\xi\\ -\bar\xi & \xi\end{pmatrix}\begin{pmatrix}I_{1\bar1} & 0\\ 0 & I_{\bar11}\end{pmatrix}\begin{pmatrix}\xi' & \bar\xi'\\ \bar\xi' & \xi'\end{pmatrix}\begin{pmatrix}\bar\gamma & 0\\ 0 & \gamma\end{pmatrix},\\
&\tilde c_-^L(l,m,s,m_0;X'';X;X')|_{m_0=M''-M,\ J''=J+\frac{k}{2},\ \alpha''=\alpha}\\
&=-\frac{ik\mu}{\pi}\mathbf{s}(J+J'-l+\tfrac{k}{2})\begin{pmatrix}\gamma & 0\\ 0 & \bar\gamma\end{pmatrix}\begin{pmatrix}\bar\xi & -\xi\\ -\xi & \bar\xi\end{pmatrix}\begin{pmatrix}I_{\bar11} & 0\\ 0 & I_{1\bar1}\end{pmatrix}\begin{pmatrix}\bar\xi' & \xi'\\ \xi' & \bar\xi'\end{pmatrix}\begin{pmatrix}\bar\gamma & 0\\ 0 & \gamma\end{pmatrix},\\
&\gamma\ =\ e^{\frac{i\pi}{2}(M-M'-m+\alpha-\alpha'-s)},\quad \xi\ =\ e^{\frac{i\pi}{2}(J-J'-l)},\quad \xi'\ =\ e^{\frac{i\pi}{2}(J-J'-l+k)}.
\end{aligned} \tag{6.40}$$

A number of recursion relations for d are obtained easily, and by solving them we find

$$\begin{aligned}
d(l,m,s;X;X') &= -i(\nu b^{2-2b^2})^{l+\frac12}\frac{\mathbf{G}(b(2l+1))}{\mathbf{G}(-b(2l+1))}\mathbf{S}(b(l+J+J'+2+\tfrac{k}{2}))\mathbf{S}(b(l+J-J'+1))\\
&\quad\times\mathbf{S}(b(l-J+J'+1))\mathbf{S}(b(l-J-J'+\tfrac{k}{2}))\\
&\quad\times\begin{pmatrix}\gamma & 0\\ 0 & \bar\gamma\end{pmatrix}\begin{pmatrix}\xi & -\bar\xi\\ -\bar\xi & \xi\end{pmatrix}\begin{pmatrix}\frac{\Gamma(1+l+m)}{\Gamma(-l+m)} & 0\\ 0 & \frac{\Gamma(1+l-m)}{\Gamma(-l-m)}\end{pmatrix}\begin{pmatrix}\tilde\xi & \bar{\tilde\xi}\\ \bar{\tilde\xi} & \tilde\xi\end{pmatrix}\begin{pmatrix}\bar\gamma & 0\\ 0 & \gamma\end{pmatrix}\\
&\xi\ =\ e^{\frac{i\pi}{2}(J-J'-l)},\quad \tilde\xi\ =\ e^{\frac{i\pi}{2}(J-J'+l+1)},\quad \gamma\ =\ e^{\frac{i\pi}{2}(M-M'-m+\alpha-\alpha'-s)}.
\end{aligned} \tag{6.41}$$

For boundary operators between B-branes, the reflection coefficient for those proportional to $\lambda,\bar\lambda$ also becomes 2×2 matrix,

$$\begin{pmatrix}\lambda B_m^{l(s)}\\ \bar\lambda B_m^{l(s)}\end{pmatrix} = \begin{pmatrix}d'^{\lambda}_{\lambda} & d'^{\lambda}_{\bar\lambda}\\ d'^{\bar\lambda}_{\lambda} & d'^{\bar\lambda}_{\bar\lambda}\end{pmatrix}\begin{pmatrix}\lambda B_m^{-l-1(s)}\\ \bar\lambda B_m^{-l-1(s)}\end{pmatrix}. \tag{6.42}$$

The calculation of the reflection coefficient $d'(l,m,s;X,X')$ proceeds in the same way as before. We only present the final result,

$$d'(l,m,s;X;X') = -i(\nu b^{2-2b^2})^{l+\frac12}\frac{\mathbf{G}(b(2l+1))}{\mathbf{G}(-b(2l+1))}\mathbf{S}(b(l+J+J'+2)\mathbf{S}(b(l+J-J'+1+\tfrac{k}{2}))$$

$$\times \mathbf{S}(b(l - J + J' + 1 + \tfrac{k}{2}))\mathbf{S}(b(l - J - J'))$$
$$\times \begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} \begin{pmatrix} \eta & \bar{\eta} \\ \bar{\eta} & \eta \end{pmatrix} \begin{pmatrix} \frac{\Gamma(1+l+m)}{\Gamma(-l+m)} & 0 \\ 0 & \frac{\Gamma(1+l-m)}{\Gamma(-l-m)} \end{pmatrix} \begin{pmatrix} \tilde{\eta} & -\bar{\tilde{\eta}} \\ -\bar{\tilde{\eta}} & \tilde{\eta} \end{pmatrix} \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} \quad (6.43)$$
$$\eta = e^{\frac{i\pi}{2}(J+J'+l)}, \quad \tilde{\eta} = e^{\frac{i\pi}{2}(J+J'-l-1)}, \quad \gamma = e^{\frac{i\pi}{2}(M+M'+m-\alpha+\alpha'+s)}.$$

The reflection relation for chiral operators can also be obtained in the same way as for A-brane case. It can be put in the following form,

$$\begin{pmatrix} \xi & \bar{\xi} \\ \bar{\xi} & \xi \end{pmatrix} \begin{pmatrix} \gamma_- & 0 \\ 0 & \bar{\gamma}_- \end{pmatrix} \begin{pmatrix} \lambda\bar{\lambda}B_l^{l(s)} \\ \bar{\lambda}\lambda B_l^{l(s)} \end{pmatrix} = d^- \cdot \begin{pmatrix} \eta & -\bar{\eta} \\ -\bar{\eta} & \eta \end{pmatrix} \begin{pmatrix} \tilde{\gamma}_- & 0 \\ 0 & \bar{\tilde{\gamma}}_- \end{pmatrix} \begin{pmatrix} \lambda B_{-\tilde{l}}^{\tilde{l}(s-1)} \\ \bar{\lambda} B_{-\tilde{l}}^{\tilde{l}(s-1)} \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} \xi & \bar{\xi} \\ \bar{\xi} & \xi \end{pmatrix} \begin{pmatrix} \gamma_+ & 0 \\ 0 & \bar{\gamma}_+ \end{pmatrix} \begin{pmatrix} \lambda\bar{\lambda}B_{-l}^{l(s)} \\ \bar{\lambda}\lambda B_{-l}^{l(s)} \end{pmatrix} = d^+ \cdot \begin{pmatrix} \eta & -\bar{\eta} \\ -\bar{\eta} & \eta \end{pmatrix} \begin{pmatrix} \tilde{\gamma}_+ & 0 \\ 0 & \bar{\tilde{\gamma}}_+ \end{pmatrix} \begin{pmatrix} \lambda B_{\tilde{l}}^{\tilde{l}(s+1)} \\ \bar{\lambda} B_{\tilde{l}}^{\tilde{l}(s+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix},$$
$$\xi = e^{\frac{i\pi}{2}(J-J'-l)} \qquad \eta = e^{\frac{-i\pi}{2}(J+J'+l)}$$
$$\gamma_\mp = e^{\frac{i\pi}{2}(M-M'\mp l+\alpha-\alpha'-s)} \qquad \tilde{\gamma}_\mp = e^{\frac{i\pi}{2}(M+M'\mp\tilde{l}-\alpha+\alpha'+s\mp 1)} \quad (6.44)$$

The right hand sides of these equations mean that a suitable linear combination of $\lambda\bar{\lambda}B_{\pm l}^{l(s)}$ and $\bar{\lambda}\lambda B_{\pm l}^{l(s)}$ has a partner under the reflection $l \leftrightarrow \tilde{l}$, while another suitable linear combination should be regarded as zero. The coefficients $d^\mp$ are given by

$$d^\mp(l,s;X;X') = \mp(\nu b^{-2b^2})^{l+\frac{1}{2}+\frac{k}{4}} \frac{\mathbf{G}(b(2l+k+1))}{\mathbf{G}(-b(2l+1))}$$
$$\times \mathbf{S}(b(l+J+J'+2+\tfrac{k}{2}))\mathbf{S}(b(l-J-J'+1+k))$$
$$\times \mathbf{S}(b(l+J-J'+1+k))\mathbf{S}(b(l-J-J'+\tfrac{k}{2})). \quad (6.45)$$

Finally, let us calculate the open string spectrum between two B-branes using boundary reflection coefficients. The spectral densities of boundary operators proportional to $(\lambda\bar{\lambda},\ \bar{\lambda}\lambda)$ or $(\lambda,\ \bar{\lambda})$ are obtained as suitable derivatives of $(\log\det d)$ or $(\log\det d')$.

$$\log\det d \sim -\int_{-\infty}^{\infty} \frac{dp}{p} \frac{e^{(2l+1)\pi p}\left\{\cosh 2\pi p(J+J'+1) + \cosh k\pi p \cosh 2\pi p(J-J')\right\}}{\sinh(\pi p)\sinh(k\pi p)},$$
$$\log\det d' \sim -\int_{-\infty}^{\infty} \frac{dp}{p} \frac{e^{(2l+1)\pi p}\left\{\cosh 2\pi p(J-J') + \cosh k\pi p \cosh 2\pi p(J+J'+1)\right\}}{\sinh(\pi p)\sinh(k\pi p)}. \quad (6.46)$$

These are in precise agreement with the spectral densities ρ_0^B, ρ_1^B of (6.38). This result also suggests that the correct normalization of the wave functions for the B-branes (4.55) is to set

$$T_0 = \sqrt{\frac{\pi}{4k}}. \quad (6.47)$$

7. Concluding remarks

We now understand the branes in $N = 2$ Liouville theory as boundary states, which are algebraic objects satisfying boundary conditions on $N = 2$ supercurrents, and also in terms of worldsheet actions containing boundary interactions. We have obtained an explicit correspondence between two descriptions, and various structure constants of the theory on the disc have been analyzed to the same extent as for the $N = 0$ and $N = 1$ Liouville theories.

The boundary interactions for B-branes proposed in this paper can be understood within the framework of Landau-Ginzburg theory, but the ones for A-branes are new. It is therefore necessary to understand the properties of these interactions from various viewpoints, such as mirror coset model.

For A-branes, the description in terms of boundary interactions is expected to apply only to those corresponding to non-degenerate representations. Some degenerate A-branes might be described by the theory on a pseudosphere (a recent work [17] has analyzed this issue). For B-branes, the relation between the labels of branes and the representations of $N = 2$ superconformal algebra is less clear.

We have not paid much attention to the open or closed string states belonging to discrete representations. Although they will not invalidate the analysis of the present paper, they will play a significant role in certain problems in string theory. It is also important to understand the modular transformation property of characters for these representations.

As a perturbed linear dilaton CFT, $N = 2$ Liouville theory has a structure very similar to the sine-Liouville theory, which is believed to be dual to the bosonic $SL(2, \mathbb{R})/U(1)$ coset model describing two-dimensional black hole. The boundary states in the sine-Liouville theory are expected to be described by a similar set of boundary interactions including boundary fermions. It would be interesting to study the D-branes in these related models and their Wick-rotated cousins along the same path.

Acknowledgments

It is a pleasure to thank T. Eguchi, J. Hashiba, K. Hori, T. Kimura, A. Pakman, D. Page and Y. Sugawara for useful discussions and comments. The early part of this work was done in collaboration with J. Hashiba and then with K. Hori. A part of the work was carried out during the author's stay at the RIMS and YITP at Kyoto university.

The work for the second version started during the collaboration with S. Ribault on related models which lead to the discovery of some fatal errors in the previous version. A part of the work was carried out during the visit to the ETH Zurich, and the author thanks the string theory group at the Institute for Theoretical Physics for hospitality and discussions.

A. Some useful formulae

In the main text we frequently used

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}, \quad \mathbf{s}(x) \equiv \sin(\pi x), \quad \mathbf{c}(x) \equiv \cos(\pi x). \tag{A.1}$$

The functions $\Upsilon, \mathbf{G}$ and $\mathbf{S}$ are defined by

$$\begin{aligned}
\log \mathbf{G}(x) &= \int_0^\infty \frac{dt}{t}\left[\frac{e^{-Qt/2}-e^{-xt}}{(1-e^{-bt})(1-e^{-t/b})}+\frac{e^{-t}}{2}\left(\frac{Q}{2}-x\right)^2+\frac{1}{t}\left(\frac{Q}{2}-x\right)\right],\\
\log \Upsilon(x) &= \int_0^\infty \frac{dt}{t}\left[e^{-2t}\left(\frac{Q}{2}-x\right)^2-\frac{\sinh^2\{(\frac{Q}{2}-x)t\}}{\sinh(bt)\sinh(t/b)}\right],\\
\log \mathbf{S}(x) &= \int_0^\infty \frac{dt}{t}\left[\frac{2x-Q}{t}-\frac{\sinh\{(x-\frac{Q}{2})t\}}{2\sinh(\frac{bt}{2})\sinh(\frac{t}{2b})}\right],
\end{aligned} \tag{A.2}$$

where $Q = b + b^{-1}$, and are characterized by the shift relations

$$\begin{aligned}
&\mathbf{G}(x+b) = \mathbf{G}(x)\frac{b^{\frac{1}{2}-bx}\Gamma(bx)}{\sqrt{2\pi}},\ \Upsilon(x+b) = \Upsilon(x)b^{1-2bx}\gamma(bx),\ \mathbf{S}(x+b) = \mathbf{S}(x)2\sin(b\pi x),\\
&\mathbf{G}(x+\tfrac{1}{b}) = \mathbf{G}(x)\frac{b^{\frac{x}{b}-\frac{1}{2}}\Gamma(\frac{x}{b})}{\sqrt{2\pi}},\ \ \Upsilon(x+\tfrac{1}{b}) = \Upsilon(x)b^{\frac{2x}{b}-1}\gamma(\tfrac{x}{b}),\ \ \mathbf{S}(x+\tfrac{1}{b}) = \mathbf{S}(x)2\sin(\tfrac{\pi x}{b})
\end{aligned} \tag{A.3}$$

Note also that

$$\Upsilon(x) = \mathbf{G}(x)\mathbf{G}(Q-x), \quad \mathbf{S}(x) = \frac{\mathbf{G}(Q-x)}{\mathbf{G}(x)}. \tag{A.4}$$

$\mathbf{G}(x)$ has poles at $x = -mb - nb^{-1}$ $(m, n \in \mathbb{Z}_{\geq 0})$ and no poles.

The functions $\eta(\tau)$ and $\vartheta(\nu,\tau)$ are defined by $(q \equiv e^{2\pi i\tau}, z \equiv e^{2\pi i\alpha})$

$$\begin{aligned}
\eta(\tau) &= q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n),\\
\vartheta(\alpha,\tau) &= \prod_{n\geq 1}(1-q^n)(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}}) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}z^n,
\end{aligned} \tag{A.5}$$

and obey the modular S transformation law:

$$\vartheta(\alpha,\tau_o) = (-i\tau_c)^{\frac{1}{2}}q_c^{\frac{\alpha^2}{2}}\vartheta(-\alpha\tau_c,\tau_c), \quad \eta(\tau_o) = (-i\tau_c)^{\frac{1}{2}}\eta(\tau_c) \quad (\tau_o\tau_c = -1). \tag{A.6}$$

In the main text we often encountered the contour integrals of the following form:

$$\begin{aligned}
&\int_{0<s<t<1} ds dt s^a(1-s)^b t^{\bar{a}}(1-t)^{\bar{b}}(t-s)^{-k-1}\\
&= \frac{\Gamma(1+a+\bar{a}-k)\Gamma(1+b+\bar{b}-k)\Gamma(a+1)\Gamma(\bar{b}+1)}{\Gamma(a-\bar{c}+1)\Gamma(\bar{b}-c+1)}\\
&\quad \times {}_3F_2(a+1,\bar{b}+1,k-c-\bar{c};a-\bar{c}+1,\bar{b}-c+1;1) \equiv G_k\left[\begin{smallmatrix} a & b & c\\ \bar{a} & \bar{b} & \bar{c}\end{smallmatrix}\right],\\
&\quad (c = k-1-a-b, \ \ \bar{c} = k-1-\bar{a}-\bar{b})
\end{aligned} \tag{A.7}$$

The function G_k satisfies the equalities

$$c+\bar{c} = k \ \ \Rightarrow \ \ G_k\left[\begin{smallmatrix} a & b & c\\ \bar{a} & \bar{b} & \bar{c}\end{smallmatrix}\right] = \frac{\Gamma(-a-\bar{a}-1)\Gamma(-b-\bar{b}-1)\Gamma(a+1)\Gamma(\bar{b}+1)}{\Gamma(-\bar{a})\Gamma(-b)}, \tag{A.8}$$

$$\mathbf{s}(a)\mathbf{s}(\bar{a})G_k\left[\begin{smallmatrix} \bar{a} & \bar{b} & \bar{c}\\ a & b & c\end{smallmatrix}\right] + \mathbf{s}(a)\mathbf{s}(k-\bar{a})G_k\left[\begin{smallmatrix} a & b & c\\ \bar{a} & \bar{b} & \bar{c}\end{smallmatrix}\right] = \mathbf{s}(c)\mathbf{s}(\bar{c})G_k\left[\begin{smallmatrix} c & b & a\\ \bar{c} & \bar{b} & \bar{a}\end{smallmatrix}\right] + \mathbf{s}(k-c)\mathbf{s}(\bar{c})G_k\left[\begin{smallmatrix} \bar{c} & \bar{b} & \bar{a}\\ c & b & a\end{smallmatrix}\right]. \tag{A.9}$$

JHEP12(2006)061

References

[1] M. Goulian and M. Li, *Correlation functions in Liouville theory*, *Phys. Rev. Lett.* **66** (1991) 2051.

[2] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B 429** (1994) 375 [hep-th/9403141].

[3] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [hep-th/9506136].

[4] J. Teschner, *On the Liouville three point function*, *Phys. Lett.* **B 363** (1995) 65 [hep-th/9507109].

[5] R.C. Rashkov and M. Stanishkov, *Three-point correlation functions in $n = 1$ super lioville theory*, *Phys. Lett.* **B 380** (1996) 49 [hep-th/9602148].

[6] R.H. Poghosian, *Structure constants in the $N = 1$ super-Liouville field theory*, *Nucl. Phys.* **B 496** (1997) 451 [hep-th/9607120].

[7] J. Teschner, *Liouville theory revisited*, *Class. and Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[8] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[9] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[10] K. Hosomichi, *Bulk-boundary propagator in Liouville theory on a disc*, *JHEP* **11** (2001) 044 [hep-th/0108093].

[11] B. Ponsot and J. Teschner, *Boundary Liouville field theory: boundary three point function*, *Nucl. Phys.* **B 622** (2002) 309 [hep-th/0110244].

[12] T. Fukuda and K. Hosomichi, *Super Liouville theory with boundary*, *Nucl. Phys.* **B 635** (2002) 215 [hep-th/0202032].

[13] C. Ahn, C. Rim and M. Stanishkov, *Exact one-point function of $N = 1$ super-Liouville theory with boundary*, *Nucl. Phys.* **B 636** (2002) 497 [hep-th/0202043].

[14] T. Eguchi and Y. Sugawara, *Modular bootstrap for boundary $N = 2$ Liouville theory*, *JHEP* **01** (2004) 025 [hep-th/0311141].

[15] C.-r. Ahn and M. Yamamoto, *Boundary action of $N = 2$ super-Liouville theory*, *Phys. Rev.* **D 69** (2004) 026007 [hep-th/0310046].

[16] C. Ahn, M. Stanishkov and M. Yamamoto, *One-point functions of $N = 2$ super-Liouville theory with boundary*, *Nucl. Phys.* **B 683** (2004) 177 [hep-th/0311169].

[17] C. Ahn, M. Stanishkov and M. Yamamoto, *Zz-branes of $N = 2$ super-Liouville theory*, *JHEP* **07** (2004) 057 [hep-th/0405274].

[18] D. Israel, A. Pakman and J. Troost, *Extended* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *characters, or modular properties of a simple non-rational conformal field theory*, *JHEP* **04** (2004) 043 [hep-th/0402085].

[19] D. Israel, A. Pakman and J. Troost, *D-branes in $N = 2$ Liouville theory and its mirror*, *Nucl. Phys.* **B 710** (2005) 529 [hep-th/0405259].

[20] A. Giveon, D. Kutasov and A. Schwimmer, *Comments on D-branes in AdS*$_3$, *Nucl. Phys.* **B 615** (2001) 133 [hep-th/0106005].

[21] A. Parnachev and D.A. Sahakyan, *Some remarks on D-branes in AdS*$_3$, *JHEP* **10** (2001) 022 [hep-th/0109150].

[22] P. Lee, H. Ooguri and J.W. Park, *Boundary states for AdS*$_2$ *branes in AdS*$_3$, *Nucl. Phys.* **B 632** (2002) 283 [hep-th/0112188].

[23] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the euclidean AdS*$_3$, *JHEP* **02** (2002) 016 [hep-th/0112198].

[24] S. Ribault and V. Schomerus, *Branes in the 2D black hole*, *JHEP* **02** (2004) 019 [hep-th/0310024].

[25] A. Fotopoulos, V. Niarchos and N. Prezas, *D-branes and extended characters in* $SL(2,\mathbb{R})/U(1)$, *Nucl. Phys.* **B 710** (2005) 309 [hep-th/0406017].

[26] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and* $N = 2$ *Liouville theory as mirror symmetry*, *JHEP* **08** (2001) 045 [hep-th/0104202].

[27] M.B. Green and N. Seiberg, *Contact interactions in superstring theory*, *Nucl. Phys.* **B 299** (1988) 559.

[28] M. Dine and N. Seiberg, *Microscopic knowledge from macroscopic physics in string theory*, *Nucl. Phys.* **B 301** (1988) 357.

[29] P. Di Francesco and D. Kutasov, *World sheet and space-time physics in two-dimensional (super)string theory*, *Nucl. Phys.* **B 375** (1992) 119 [hep-th/9109005].

[30] W. Boucher, D. Friedan and A. Kent, *Determinant formulae and unitarity for the* $N = 2$ *superconformal algebras in two-dimensions or exact results on string compactification*, *Phys. Lett.* **B 172** (1986) 316.

[31] S.K. Nam, *The kac formula for the* $N = 1$ *and the* $N = 2$ *superconformal algebras*, *Phys. Lett.* **B 172** (1986) 323.

[32] T. Fukuda and K. Hosomichi, *Three-point functions in sine-Liouville theory*, *JHEP* **09** (2001) 003 [hep-th/0105217].

[33] S. Govindarajan, T. Jayaraman and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, *Nucl. Phys.* **B 580** (2000) 519 [hep-th/9907131].

[34] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[35] K. Hori, *Linear models of supersymmetric D-branes*, hep-th/0012179.

[36] S. Hellerman, S. Kachru, A.E. Lawrence and J. McGreevy, *Linear sigma models for open strings*, *JHEP* **07** (2002) 002 [hep-th/0109069].

[37] A. Kapustin and Y. Li, *D-branes in Landau-Ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296]; *Topological correlators in Landau-Ginzburg models with boundaries*, *Adv. Theor. Math. Phys.* **7** (2004) 727 [hep-th/0305136]; *D-branes in topological minimal models: the Landau-Ginzburg approach*, *JHEP* **07** (2004) 045 [hep-th/0306001].

[38] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, *JHEP* **11** (2006) 043 [hep-th/0305133].